CONSTANT ABSOLUTE VORTICITY TRAJECTORIES

by R. Maine & C. Pierrehumbert

Central Meteorological Bureau, Melbourne

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Abstract: A graphical method of solution of the equation of conservation of absolute vorticity is produced and the result applied to the 500 mb constant pressure surface. A more readily applicable though approximate mathematical alternative solution is also obtained.

Forecasts made from the results of the solution of the equation at a fixed point are subjected to a verification test. These results are examined for their degree of significance. The limited number of forecasts made are found to present good agreement with actual conditions when caution is exercised in the selection of an initial wind.

1. INTRODUCTION

In the U.S. "Constant Absolute Vorticity Trajectories" have been used as an aid to prognosis since about 1942. This paper describes an attempt to apply the theory to the Australian Region without the use of mechanical aids, such as the so-called "Wiggle Waggon" designed by Wobus or the slide rule devised by Bellamy (1945).

2. THEORETICAL ASPECTS

The application of the theory relating to the conservation of absolute vorticity of a particle requires the use of many restrictive assumptions, the most objectionable being that of non-divergence. For this reason forecasts were only attempted on the 500 mb. surface as this was considered to be near the level of non-divergence. An assumption of constant speed must also be incorporated for the sake of mathematical ease.

Using the above restrictions and also one of no shear it can be seen that if absolute vorticity is to be conserved

$$K V + f = K_0 V_0 + f_0 \qquad (1)$$

where the subscripts refer to initial conditions, K is the magnitude of the curvature of a streamline, V the wind speed, and f the coriolis parameter relating to the latitude of the particle at the particular instant. For convenience an initial point was chosen such that K_o was zero (e.g. a point of inflection on the westerly waves) then, if V is constant and equal to V_o, and R is the radius of streamline curvature,

$$R = \frac{V_o}{2\Omega(\sin \phi - \sin \phi_o)} \qquad (2)$$

Given V_o and ϕ_o the above function was plotted, the graph for latitude 32° south and a range of speeds being reproduced in figure 1.

It must be noted that K is positive for cyclonic curvature and negative for anticyclonic curvature, and for the sake of mathematical ease V was assumed not to vary with time. In drawing the trajectory an attempt was made to pay attention to the no shear condition at least at the initial point if not over the whole trajectory. For this reason an initial wind was not used for a forecast unless it was relatively constant both in magnitude and direction over a period of a day or so. With this in mind an element was projected from the initial point in increments of one latitude degree in the direction given by the curvature at the latitude of the mid point of the increment and with the speed of the wind at the initial point. Great circles approximate to straight lines in middle latitudes on the conical map projection used, so the effect of the earth's curvature was taken into account. The result finally obtained, although a little tedious, was a wave asymmetrical about the initial latitude ϕ_o, although Starr (1942) by assuming no variation in the cosine of ϕ over the amplitude of the wave obtained a symmetrical trajectory.

Figure 2 represents an example of two characteristic trajectories, one with initial wind 190°/56K and with construction shown, and the other with initial wind 230°/31K with construction not shown. At places where the trajectory curved cyclonically it was assumed that a trough would occur at the time a particle originally at the initial point reached this position. The reasoning was similar for regions of anticyclonic curvature where occurrence of ridges was expected. By drawing a number of trajectories to pass through the forecast area a reasonable forecast of the predominating pattern over the forecast area could be deduced. The approach normally adopted is to draw a trajectory to pass through the forecast area on consecutive dates before the forecast date, the evidence thus being cumulative.

We now turn to a mathematical, though more approximate, solution. Since $K = d\psi/ds$ and $K_o = 0$ equation (1) reduces to

$$\frac{d\psi}{dt} + f = f_o \qquad (3)$$

where ψ is the angle made by the tangent to the streamline and the latitude circle. Making use of Rossby's assumption $f \simeq f_o + \beta y$ for a small displacement in the meridional direction y, and also the equalities

$$\frac{d\psi}{dt} = \frac{dv}{dt}/u \doteq \frac{du}{dt}/v$$

resulting from $\sin\psi = v/V$ and $\cos\psi = u/V$, we obtain the following

$$\frac{dv}{dt} + \beta_o uy = 0 \qquad (4)$$

$$-\frac{du}{dt} + \beta_o vy = 0 \qquad (5)$$

where u and v are zonal and meridional components of the constant speed $V, \beta = \beta_o = 2\,\Omega \cos\phi/a$ (assumed not to vary appreciably within the amplitude of the wave), and a the radius of the earth.

Integrating equation (5) with respect to time and using the boundary condition y = o when $u = u_o$ we find when v = o

$$y = \pm \left(\frac{2\,(V-u_o)}{\beta_o}\right)^{\frac{1}{2}} \qquad (6)$$

which is the amplitude of the periodic motion.

To obtain a formal solution of the period, combination of equations $u \doteq u_o + \frac{1}{2}\beta_o y^2$ (general form of equation (6)) and $u^2 = V^2 - v^2$ results in

$$v^2 = V^2 - u_o^2 - \beta_o u_o y^2 - \beta_o^2 y^4/4$$

or

$$\left(\frac{dy}{dt}\right)^2 = v_o^2 - \beta_o u_o y^2 - \beta_o^2 y^4/4$$

This expression yields for the time of maximum y

$$t = \int_0^{2\frac{(V - u_0)^{\frac{1}{2}}}{\beta_0}} \frac{dy}{\left(v_0^2 - \beta_0 u_0 y^2 - \frac{\beta_0^2 y^4}{4}\right)^{\frac{1}{2}}}$$

This integral is of the elliptic type for which a series solution may be found but was not investigated as a different approach gave an approximate solution of the wavelength. It should be noted that if the complete expression for β is substituted in (4) and (5) the end point is a result which neglecting third and higher powers of $\frac{y}{a}$ provides us with the solution as above.

A simpler approach to the problem is to superimpose a small perturbation on a zonal current of strength V. Using then the relations $u = U + u'$, $v = v'$ simplification of equation (4) by neglecting the term $u'y$ results in

$$\frac{d^2y}{dt^2} + \beta_0 Uy = 0 \qquad (7)$$

This has the general solution

$$y = A \sin \; (\beta_0 U)^{\frac{1}{2}} \, t$$

A being the amplitude of the motion which can be obtained from equation (6)

When $y = A$, $t = \frac{\tau}{4}$ where τ is the period

and $\tau \approx \frac{2\pi}{(\beta_0 V)^{\frac{1}{2}}}$ if $U \approx V$

Since the speed of the current is V the wavelength of the motion is

$$\lambda \approx 2\pi \left(\frac{V}{\beta_0}\right)^{\frac{1}{2}}$$

The parameters A and λ can be used to sketch the trajectory, and graphs for obtaining them at latitude 38°S are reproduced in figures 3 and 4.

A test to determine the extent to which the above expressions could be employed was made by comparing trajectories drawn by the incremental method and by use of the amplitude and wavelength functions above. It was concluded that the results did not differ significantly unless the flow deviated by more than 45° from a westerly direction.

TABLE 1

Initial Date	Initial Wind	Wavelength		Amplitude (Deg Lat)	
		Incremental Method	Sinusoidal Approx.	Incremental Method	Sinusoidal Approximation
14/5/53	250/80	101°	105°	5.8	4.5
11/7/53	210/40	61°	75°	10.0	9.4
15/5/53	190/56	59°	89°	14.6	14.2
28/5/53	240/41	75°	76°	5.6	5.0
14/3/54	230/31	62°	66°	5.8	5.7
20/7/53	330/66	78°	95°	12.4	12.0

3. REVIEW OF RELEVANT LITERATURE

A rather serious objection to the preceding arguments can be based on the fact that K_S, the streamline curvature, was assumed identical with K_T, the modulus of the horizontal curvature of the trajectory. The two are related by the following

$$K_T = K_S + \frac{1}{V}\frac{\partial \psi}{\partial t}$$

where $\frac{\partial \psi}{\partial t}$ is the local angular rate of turning of the tangent to the streamlines. Grant (1953) discusses the relative importance of the term $\frac{1}{V}\frac{\partial \psi}{\partial t}$ as applied to Rossby waves, and arrives at the conclusion that drawn and actual constant absolute vorticity trajectories can be completely out of phase when the system departs from steady state conditions. We agree with Fultz (1945) that the assumptions seem admissable in so far as the results justify them.

Fultz discusses the effect of neglecting certain of the terms in the complete expansion of $K_T V$. In particular if the shear term increases cyclonically the effect is to cause an increase in anticyclonic curvature. In this article however no attempt has been made to assess the magnitude of the shear term in the vicinity of an initial point, it being eliminated in the choice of such a point.

4. DISCUSSION OF DATA

Points of origin used to draw the trajectories were Amsterdam Island, Guildford and Cocos Island. Cocos Island as origin did not prove successful, the largest errors probably arising from the assumptions of non-divergence, and the presence of mainly zonal winds producing flat trajectories. In addition the polyconic base charts used introduced a certain amount of curvature to a great circle at low latitudes.

TABLE 2

Initial Station and Date	Period in dys from initial date to final forecast date	Feature over forecast area	Feature observed over forecast area	Deviation of axis of forecasted feature from observed feature + ve to east and -ve to west
Col.1.	Col.2.	Col.3.	Col.4.	Col.5.
AMSTERDAM				
11/11/53	1	Trough	Trough	+ 5°
18/11/53	1	T	T	0°
14/3/54	1	T	T	0°
14/5/53	1	T	T	0°
11/7/53	1	T	T	+ 5°
15/5/53	1	T	T	+ 5°
8/8/53	1½	T	T	+ 5°
13/8/53	1½	T	T	+ 5°
14/5/53	2	Ridge	Ridge	+ 5°
11/7/53	2½	R	T	
2/7/54	2½	R	R	-10°
7/7/54	2½	R	R	0°
15/5/53	2½	R	R	0°
8/8/53	3	R	R	0°
13/8/53	3	R	R	+ 5°
11/11/53	3	R	R	+15°

TABLE 2 CONTINUED

Col.1.	Col.2.	Col.3.	Col.4.	Col.5.
26/11/53	3	Trough	Trough	0°
14/3/54	3	R	R	0°
3/7/54	3	R	R	0°
4/7/54	3	T	Closed Low	0°
20/7/53	3½	T	T	+ 5°
2/7/54	3½	T	T	-15°
26/11/53	4	P	R	0°
11/7/53	4	T	T	0°
20/7/53	4½	R	R	0°
15/5/53	4½	T	Closed Low	0°
8/8/53	5	T	Closed Low	+ 5°
13/8/53	5	T	T	+ 5°
11/11/53	5	T	T	0°
18/11/53	5	T	Negligible pressure gradient	
14/3/54	5	T	T	- 5°
3/7/54	5	T	T	0°
7/7/54	5½	R	R	0°
4/7/54	5½	R	High	-10°
15/5/53	6	R	R	0°
11/7/53	6½	R	R	0°
8/8/53	7	R	R	+10°
13/8/53	7	R	R	-10°
14/3/54	7	R	R	-10°

A presentation of the results of the forecasts is given in Table 2. An estimate of the standard deviation of the errors in the positions of ridges and troughs was made. Generally the standard deviation for ridges was about 7° and the deviation for troughs about 4° of latitude. A tentative deduction from the fact that twice the sum of these deviations was less than the average distance between trough and ridge lines, would be that the above results were better than chance. However the period of the forecasts was not considered.

5. CONCLUSION

Although the results of the above treatment are encouraging it is not possible to draw any definite conclusions with the amount of data available at present. The authors hope to carry out further work on this subject.

6. ACKNOWLEDGEMENTS

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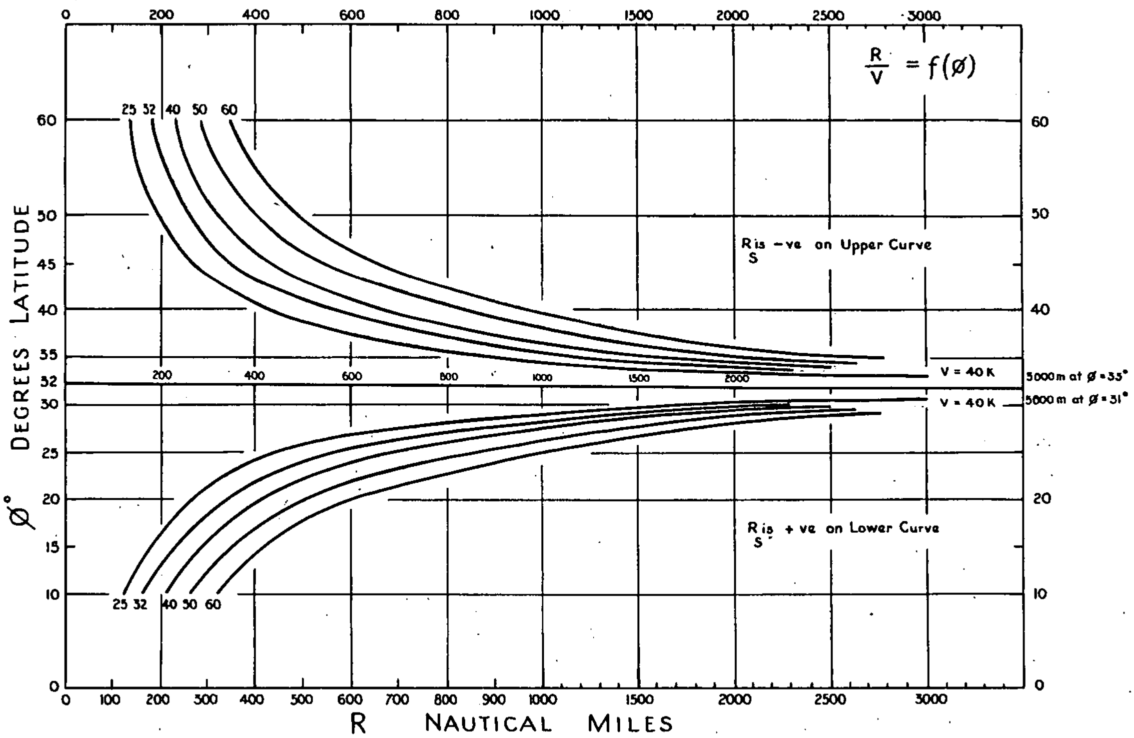

FIG.I. NOMOGRAM FOR DETERMINATION OF CURVATURE OF TRAJECTORY FOR LATITUDE 32°S.

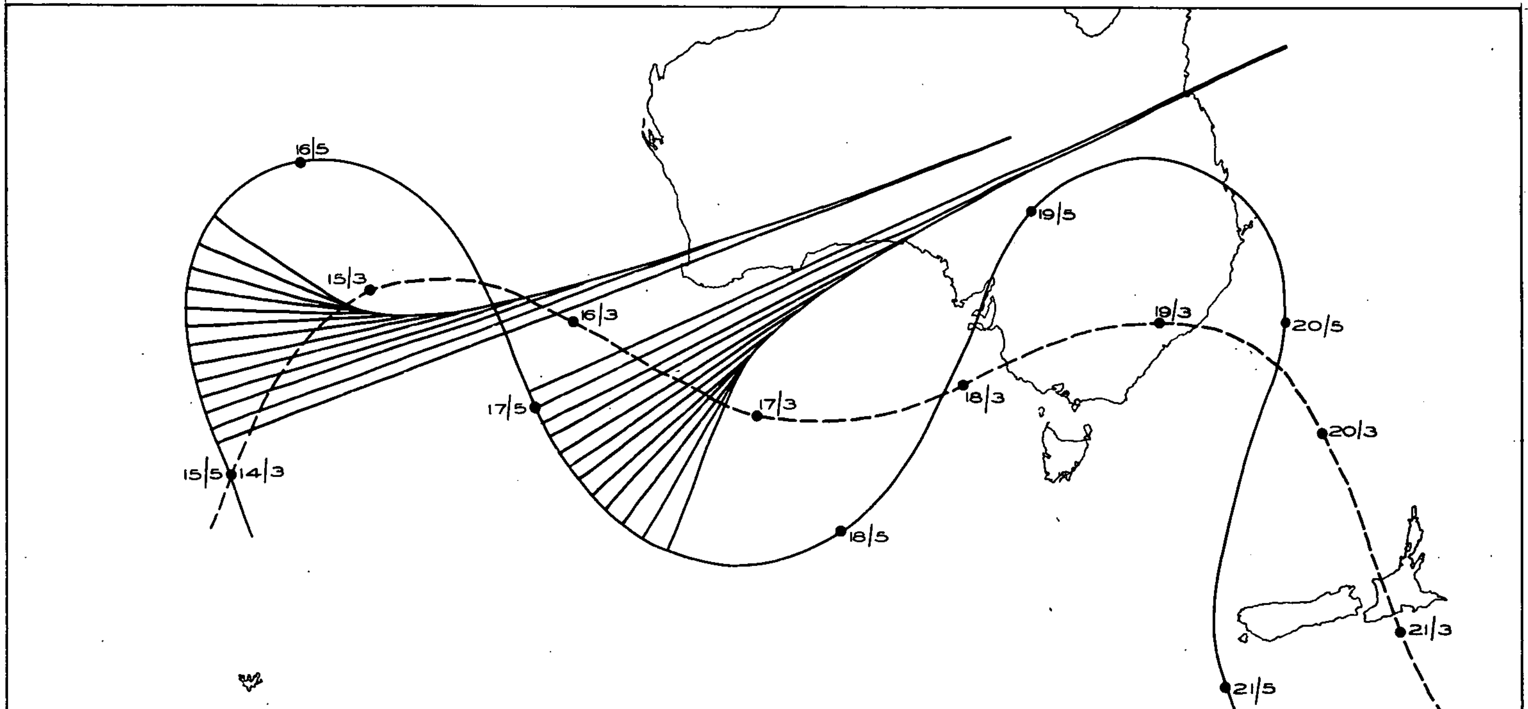

FIG. 2. —C.A.V. TRAJECTORY FOR 15-5-53 – 21-5-53 WIND 190° 56 KNOTS AT AMSTERDAM. IS. ON 15-5-53 SHOWING CONSTRUCTION.
– –C.A.V. TRAJECTORY FOR 14-3-54 – 22-3-54 WIND 230° 31 KNOTS AT AMSTERDAM. IS ON 14-3-54. CONSTRUCTION NOT SHOWN.

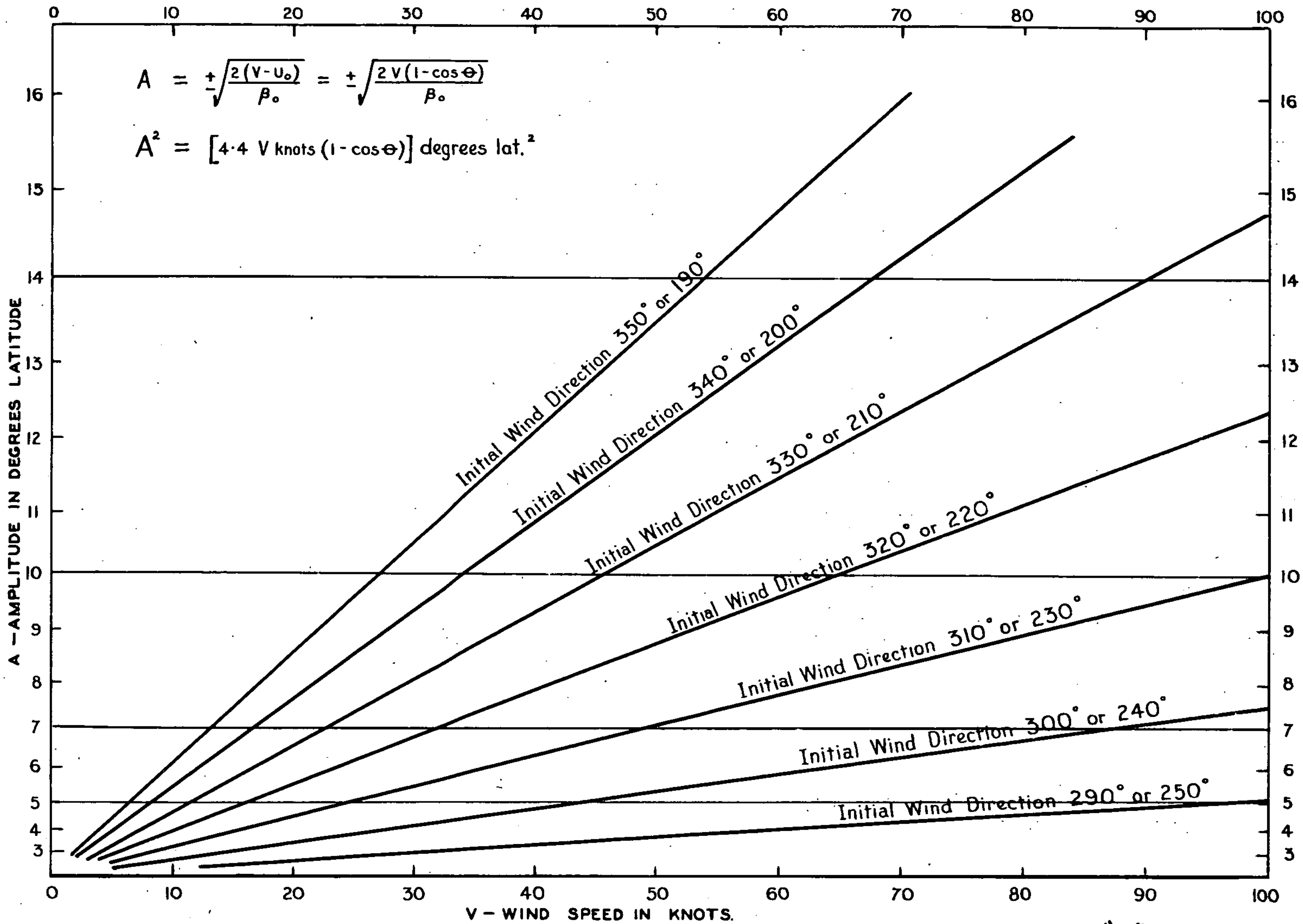

FIG. 3. NOMOGRAM FOR DETERMINATION OF AMPLITUDE "A" FOR LATITUDE 38°S.

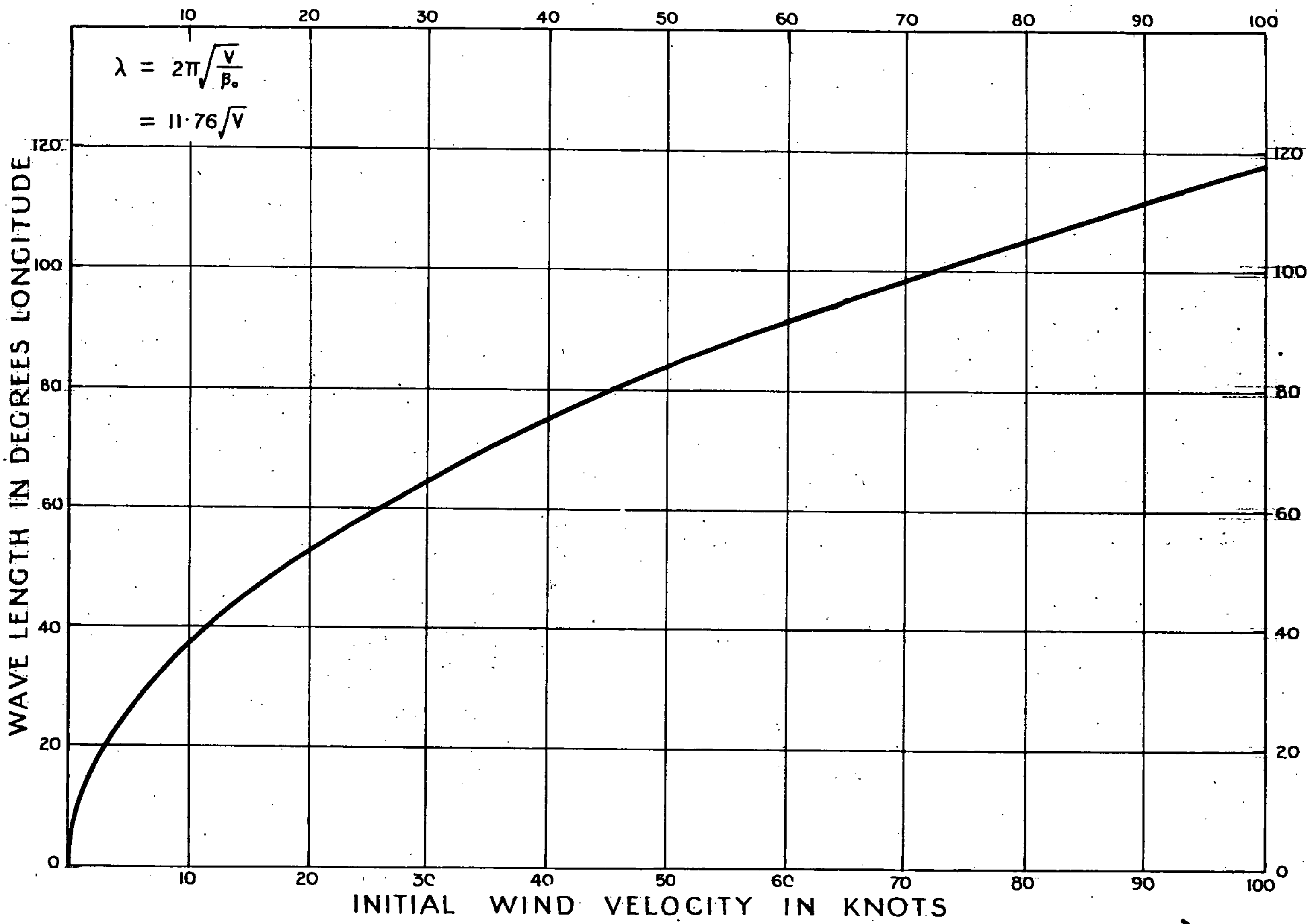

FIG. 4. NOMOGRAM FOR DETERMINATION OF WAVE LENGTH λ FOR LATITUDE 38°S.