EVALUATION OF UPPER AIR PROGNOSIS
BY ROSSBY METHOD

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Abstract: An experiment on the application of Rossby's "long wave" formula to forecasting of 24 hours displacements of troughs at the 500 mb level, has been made for the Australian region.

Rossby's simplifying assumptions and the theoretical basis of his method are discussed. Also some conclusions of U.S.A. meteorologists regarding the practical application of the formula are included in the article.

It is found that the formula is of little practical value for the relatively limited area of the southern hemisphere covered by Australian upper level charts.

1. INTRODUCTION

500 mb level prognostic charts are being prepared regularly by Central Analysis Section for 12 hours ahead. The method of preparation is based on

(i) Extrapolation

(ii) Surface prognostic charts and thickness patterns

(iii) Experience

(iv) Application of some relevant qualitative rules (Scherhag, Sutcliffe, Petterssen).

The analyst preparing 500 mb prognostic charts in Australia faces

(i) Absence of upper level observations over large areas west and south of the Australian continent.
(ii) Few observations over the Tasman Sea, Coral Sea and South Pacific area.

(iii) Lack of systematic investigations of upper level flow over large subtropical and tropical areas of the southern hemisphere and almost complete absence of upper level charts from middle and higher latitudes of the hemisphere.

Many attempts have been made in the past two decades in the northern hemisphere to increase the reliability of upper level prognostic charts by application of "objective" formulae obtained theoretically from the equations of motion. These formulae may help to compute the rate of displacement of troughs, ridges, lows, highs or, by using an electronic computer, to draw upper level prognostic charts up to 48-72 hours ahead.

To evaluate the application of "objective" formula to the prognostication of displacements of troughs on 500 mb level in Australia Rossby’s method has been examined for the period 15 September to 1 November, 1956.

It is useful to consider first the basis of Rossby’s theory.

2. THEORETICAL BASIS OF ROSSBY’S FORMULA

To compute quantitatively the approximate speed of the displacement of a "long wave" in upper westerlies from the hydrodynamical equations, Rossby (1939) made a number of simplifying assumptions regarding the characteristics of the atmosphere as a fluid and the dynamics and kinematics of the flow.

Rossby’s assumptions were:

(i) Movement of the atmosphere is horizontal and non-divergent.

(ii) The flow is frictionless.

(iii) Density of air is constant in the horizontal.

(iv) W-E component of wind is constant in time and space.
(v) Motion on an upper level surface is "nearly balanced" (at any time isobars and streamlines "nearly" coincide).

(vi) Choosing \( x \) axis of a Cartesian coordinate system (with the origin fixed at latitude \( \phi \)) towards the east and \( y \) axis towards the north (in the northern hemisphere) Rossby assumes that on the constant zonal current \( u \) (assumption (iv)) a sinusoidal disturbance is superimposed represented by the S\-N component of the wind (\( y \) component)

\[
u = v_0 \cos \frac{2\pi}{L} (x - ct)
\]

where \( v_0 \) is the amplitude of disturbance independent of \( y \), \( L \) the wave length and \( c \) the velocity of the displacement of this wave along \( x \) axis.

It is assumed that \( c \) is constant in time and is the same in all points in the \( xy \) plane.

(vii) Application of Cartesian coordinate system means that curvature of the earth's surface over the area covered by the long wave is neglected.

All assumptions made should be as near as possible to real flow and must satisfy the hydrodynamical equations in Cartesian coordinate system:

\[rac{d\nu}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - 2\omega \sin \phi \nu \quad \ldots \quad (1)
\]

\[rac{d\nu}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + 2\omega \sin \phi \nu
\]

(where \( \phi \) is the latitude of the origin of the coordinate system) and the continuity equation for a fluid with constant density:

\[rac{\partial \nu}{\partial x} + \frac{\partial \nu}{\partial y} = 0 \quad \ldots \quad (2)
\]

If assumptions (iv) and (vi) do not satisfy equations (1) and (2) the assumed distribution of wind is not possible dynamically and kinematically respectively.
If \( u = \text{const.} = u_0 \) and \( v = v_0 \cos \frac{2\pi}{L} (x - ct) \), all derivatives of \( u \) in time and space are zero or
\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial c} = 0
\]
and, as \( v \) is independent of \( y \), \( \partial v/\partial y = 0 \).

It is seen that the continuity equation is satisfied and the flow is possible kinematically.

Representing \( \frac{dv}{dt} \) in the form \( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \)
we find
\[
\frac{dv}{dt} = v_0 e^{\frac{2\pi}{L} \sin \frac{2\pi}{L} (x - ct)} - u_0 v_0 \frac{2\pi}{L} \sin \frac{2\pi}{L} (x - ct)
\]
and substituting values of \( u \) and \( v \) into equation (1) one obtains
\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} - 2 \omega \sin \psi v_0 \cos \frac{2\pi}{L} (x - ct)
\]
\[
(c - u_0) v_0 \frac{2\pi}{L} \sin \frac{2\pi}{L} (x - ct) = -\frac{1}{\rho} \frac{\partial p}{\partial y} + 2 u_0 \omega \sin \psi
\]

Therefore
\[
\frac{1}{\rho} \frac{\partial p}{\partial x} = -2 \omega \sin \psi v_0 \cos \frac{2\pi}{L} (x - ct)
\]
\[
\frac{1}{\rho} \frac{\partial p}{\partial y} = -(c - u_0) v_0 \frac{2\pi}{L} \sin \frac{2\pi}{L} (x - ct) + 2 \omega \sin \psi v_0
\]

\[\text{...... (3)}\]

If \( \rho \) is constant, equations (3) are compatible only if the partial derivatives of the first with respect to \( y \) and of the second with respect to \( x \) are identical i.e.
\[
\frac{1}{\rho} \frac{\partial p}{\partial y} \left( \frac{\partial p}{\partial x} \right) \equiv \frac{1}{\rho} \frac{2}{\partial x} \left( \frac{\partial p}{\partial y} \right)
\]

hence
\[
\frac{2}{\partial y} \left[ 2 \omega \sin \psi v_0 \cos \frac{2\pi}{L} (x - ct) \right]
\]
\[
\equiv (c - u_0) v_0 \frac{2\pi}{L} \frac{\partial}{\partial x} \sin \frac{2\pi}{L} (x - ct) + \frac{\partial}{\partial x} 2 \omega \sin \psi v_0 u_0
\]
but \[ \frac{\partial}{\partial x} 2 \omega \sin \varphi = 0 \]

and \[ 2 \omega v_0 \omega \frac{3 \pi}{L} (x-c t) \frac{\partial \sin \varphi}{\partial y} \equiv (C-u_0) v_0 \frac{4 \pi^2}{L^2} \cos \frac{2 \pi}{L} (x-c t) \]

and finally \[ 2 \omega \frac{\partial \sin \varphi}{\partial y} \equiv (C-u_0) \frac{4 \pi^2}{L^2} \]

\[ \ldots (4) \]

\( c, u_0, \pi, L \) on the right side are constant and therefore the whole right side is constant. The left side \( 2 \omega \frac{\partial \sin \varphi}{\partial y} \) obviously is not constant, if \( \varphi \) is considered as a variable latitude (which is not the case for a fixed coordinate system). Rossby assumes that

\[ \delta y = r \delta \varphi \quad \text{and} \quad \frac{\partial \sin \varphi}{\partial y} = \frac{\cos \varphi}{r} \]

In this case it means that strictly speaking the distribution of wind assumed by Rossby is dynamically impossible and the pressure distribution which would correspond to the motion assumed cannot exist.

But Rossby made the assumption that \( 2 \omega \frac{\partial \sin \varphi}{\partial y} = \beta \) is constant and therefore assumed that the motion characterised by assumptions (iv) and (vi) is dynamically possible.

Introducing

\[ \frac{\partial}{\partial y} 2 \omega \sin \varphi = \beta = \text{const.} \]

we obtain by simple integration:

\[ 2 \omega \sin \varphi = \beta y + 2 \omega \sin \varphi_0 \]

where \( \varphi_0 = \text{const.} \)

Rossby remarked that last assumption is justified for equatorial areas where \( \beta \) changes slowly with the latitude but at latitude 60° \( \beta \) is about 30 per cent less than its value at 45°.

Assuming \( \beta = \text{const.} \), we obtain from (4):

\[ \beta = (C-u_0) \frac{4 \pi^2}{L^2} \quad \text{and} \quad C-u_0 = \frac{L^2 \beta}{4 \pi^2} \quad \ldots (5) \]
This is the well known Rossby formula for the speed of displacement of "long waves". Measuring wave length \( L \) (distance between axes of two successive upper troughs) and knowing the constant W-E component of wind \( u_0 \) on an upper level isobaric (or equipotential) surface we can compute \( c \) — the speed of displacement of the trough. Assuming that \( c \) will remain constant in the next 24 hours we can calculate the longitudinal position of the axis of the trough on a 24 hours prognostic chart.

Assuming \( \beta = \text{const.} \) and \( 2\omega \sin \phi = \beta y + 2\omega \sin \varphi_0 \) we can easily obtain equations of isobars by direct integration of equations (3):

\[
\frac{\partial \beta}{\partial x} = -(\beta y + 2\omega \sin \varphi_0) \frac{\sin \frac{2\pi}{L}}{L} (x - ct)
\]

\[
\frac{\partial \beta}{\partial y} = -\frac{\omega^2}{4\pi^2} \frac{\sin \frac{2\pi}{L}}{L} (x - ct) + (\beta y + 2\omega \sin \varphi_0) u_0
\]

Integrating first equation in respect to \( x \) regarding \( y \) and \( t \) as constants we obtain:

\[
\frac{\partial}{\partial x} (x, y, t) = -(\beta y + 2\omega \sin \varphi_0) \frac{\sin \frac{2\pi}{L}}{L} (x - ct) + F(y, t)
\]

where \( F(y, t) \) is an arbitrary function of \( y \) and \( t \), independent of \( x \).

Substituting this expression in the second equation we have:

\[
\frac{\partial F}{\partial x} = (\beta y + 2\omega \sin \varphi_0) u_0
\]

\[
F(y, t) = (\frac{1}{2} \beta y^2 + 2\omega \sin \varphi_0 y) u_0 + \psi(t)
\]

where \( \psi(t) \) is an arbitrary function of time independent of \( x \) and \( y \).

We may assume \( \psi(t) = \frac{1}{2} \beta \rho_c \) where \( \rho_c \) is the constant pressure value at the origin of the coordinate system chosen at time \( t = 0 \).

Finally we obtain:

\[
= (\beta y + 2\omega \sin \varphi_0) \frac{\sin \frac{2\pi}{L} (x - ct)}{L} + \left( \frac{1}{2} \beta y^2 + 2\omega \sin \varphi_0 \right) u_0
\]

\[
\frac{\partial}{\partial x} \left[ F(x, y, t) - \rho_c \right]
\]

\[
= (\beta y + 2\omega \sin \varphi_0) \frac{\sin \frac{2\pi}{L} (x - ct)}{L} + \left( \frac{1}{2} \beta y^2 + 2\omega \sin \varphi_0 \right) u_0
\]

\[
\cdots \cdots \cdots \cdots (6)
\]
On the other hand the equation of streamlines may be written in the form

\[ \frac{dx}{u} = \frac{dy}{v_0 \cos \frac{2\pi}{L}(x-ct)} \]

After integration

\[ y = y_0 + \frac{v_0}{u_0} \frac{L}{2\pi} \sin \frac{2\pi}{L}(x-ct) \] \[ \ldots \ldots (7) \]

This equation represents a family of streamlines at any time \( t \) as parallel sinusoids with the amplitude \( \frac{v_0}{u_0} \frac{L}{2\pi} \) and wave length \( L \).

Comparing (6) and (7) one can conclude that isobars and streamlines do not coincide completely. The equation of an isobar corresponding to a constant pressure \( \frac{\rho}{\rho_0} = \rho \), is:

\[ \frac{\rho}{\rho_0} (\rho - \rho_0) = \text{const} \]
\[ = -(\beta y + 2\omega \sin \varphi_0 v_0 \frac{L}{2\pi} \sin \frac{2\pi}{L}(x-ct) + (\frac{\rho}{\rho_0} - 1) y) u_0 \]

Taking in account that \( \beta \) is of the order \( 10^{-13} \) and \( 2\omega \sin \varphi_0 \) is of the order \( 10^{-4} \) we obtain, with a reasonable approximation:

\[ \frac{\rho}{\rho_0} (\rho - \rho_0) = \text{const} = - 2\omega \sin \varphi_0 v_0 \frac{L}{2\pi} \sin \frac{2\pi}{L}(x-ct) + 2\omega \sin \varphi_0 v_0 y \]

or

\[ y = \frac{\text{const}}{2\omega \sin \varphi_0 u_0} + \frac{v_0}{u_0} \frac{L}{2\pi} \sin \frac{2\pi}{L}(x-ct) \]

which is equation of a sinusoid of amplitude \( \frac{v_0 L}{u_0 2\pi} \) and wave length \( L \) identical with the amplitude and wave length of the streamline above. It means that isobars and streamlines "nearly" coincide and the assumption (v) is justified.

We consider now Rossby's other assumptions.

Petterssen (1956) made the following remark about these assumptions: "In reality, the assumptions are not sufficiently accurate for the quantitative
determination of the wave speed \ldots. Qualitatively, however, the relation between contour lines, isotherms and speed of propagation is in good agreement with observation, and as a first (and rough) approximation it is clearly recognisable". In particular about assumption (iv) Petterssen writes: "This assumption is at variance with observation and is likely to introduce errors larger than those arising from the assumption of non-divergence".

The hydrodynamical equations used by Rossby in the form (1) are equations of motion in a Cartesian co-ordinate system tangential to the earth's surface (or to an equipotential surface) at a fixed point at latitude $\varphi$. Application of these equations to the study of large scale motion (of the order 1000 km or more in any horizontal direction) is not justified theoretically. Curvature of the earth or equipotential surfaces cannot be disregarded on large distances.

Haurwitz (1941) pointed out that application of the equations in the spherical coordinate system (which takes into account the earth's curvature) gives results similar to Rossby's qualitatively but different quantitatively.

Haurwitz's solution of the problem of "long waves" in a spherical curvilinear coordinate system is published in Compendium of Meteorology (1951).

C.L. Godske, T. Bergeron, J. Bjerknes, R.G. Bundgaard (1957, pp 245-246 and p 351) mentioned that \ldots "only phenomena less than 1000 km in extent can be treated with validity by Cartesian coordinates".

J. Holmboe, G.F. Forsythe and W. Gustin (1945, pp 173-174) describing the "natural" and "standard" (fixed Cartesian) coordinate system state: "Hence they are valid only in the immediate neighbourhood of the origin of co-ordinates".

Rossby's consideration of $\varphi$ as a variable in the Cartesian coordinate system with the origin fixed at latitude $\varphi$ is very questionable. $\varphi$ being considered as the latitude of the fixed origin of the coordinate system (which does not change its position) cannot be a function of $y$ in this system. Therefore

$$ \frac{\partial \sin \varphi}{\partial y} \equiv 0 \quad \text{and} \quad \varphi \equiv (c-u_0) \frac{4\pi}{L} $$

(see (4))
and the motion assumed by Rossby is dynamically possible only if \( C = \omega \), i.e. the velocity of the wave is equal to the constant velocity \( \omega \) of the W-E flow. But such a conclusion, although correct theoretically, is useless in practice.

Therefore the Rossby formula should be considered as an empirical rather than a theoretical one and its value estimated only statistically by direct application to upper level charts.

H. Riehl and others (1952) using Rossby's formula arrived at some purely experimental conclusions regarding practical application of the formula. The most important conclusions are:

1. "It is usually sufficient in forecasting for North America to calculate the profile and consider the long waves in the longitude belt 20\(^\circ\)E. to 110\(^\circ\)-120\(^\circ\)E via North America".

2. "For the best results, the zonal wind speed should be that at 600 mb. Since, in general charts are drawn at 700 mb or 500 mb, we must extrapolate the winds at those levels to 600 mb".

3. "Calculation of the zonal wind speed should be made with the average value of the 600 mb zonal wind speed over a belt 10\(^\circ\) latitude wide centred on the latitude of maximum westerlies".

4. "The forecast is most reliable when a definite west-wind maximum exists somewhere in middle latitudes. When the profile is nearly uniform over most latitudes with only minor protrusions, the calculation should not be attempted. It is an oddity that Rossby's formula was derived for a zonal current that is independent of latitude, but that the application is most successful when the zonal wind is peaked".

Staff Members, Weather Forecasting Research Centre, University of Chicago (1956) experimentally forecast the displacement of 500 mb troughs and ridges in U.S.A. and computed 54 cases of displacement of "long waves" using Rossby's formula.
They found:

"The result of computation was considered as the movement of the downstream trough, rather than the mean motion of the upstream and downstream troughs. Attempts to take in to account the motion of the upstream trough gave worse results."

It appears from the above that there is little prospect of substantial help from the Rossby's formula in the Australian region since the practical requirements quoted above, rarely could be met in the Australian area:

(i) It is not always possible to locate the latitudinal belt of maximum westerlies at 500 mb as it is frequently south of 35°S, where there are no upper level observations.

(ii) The longitudinal range over which computations could be made in Australia is at best 80°, but frequently does not exceed 60°-70° (110°E to 170°-180°E).

3. RESULTS OF TESTS OF FORMULA

In most cases long waves over the Australian region consist of two troughs: one (upstream) somewhere between longitudes 110° and 130°E and the second (downstream) between longitudes 150°-175°E with a ridge between them. Not always is it possible even to decide whether the system is "long wave" or "short wave". In any case the most important forecast for Australia is the displacement of the western (upstream) not the eastern (downstream) trough. For this reason computation has been made only for upstream troughs.

The 0200 GMT 500 mb charts from 15 September to 1 November, 1956, analyzed by the Central Analysis Section of the Australian Bureau of Meteorology have been used for computation.

\( \mathcal{U}_0 \) has been computed as the geostrophic wind corresponding to average 200 feet contour spacing along the belt of the core of the stream 10° of latitude wide and extending from the axis of the western trough to the axis of the nearest well expressed eastern trough. The values of \( \mathcal{U}_0 \) and \( L \beta \phi_4 \) in terms of degrees of longitude per 24 hours were obtained from Smithsonian Meteorological Tables (1951).
The computed displacements of troughs were compared with actual displacements in following 24 hours. Results of the computation are given below and also diagramatically in Figure 1.

<table>
<thead>
<tr>
<th>No</th>
<th>Percent of all cases</th>
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<tbody>
<tr>
<td>47</td>
<td>100</td>
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</table>

<table>
<thead>
<tr>
<th>Number of charts investigated</th>
<th>47</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of cases when the formula could not be used (no long waves on the chart)</td>
<td>28</td>
<td>59</td>
</tr>
<tr>
<td>Number of cases when the formula could be used</td>
<td>19</td>
<td>41</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Percent of computed cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>----------------------------</td>
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<tr>
<td>Number of cases when computed position of the trough deviated from the real position no more than ± 4° of longitude in 24 hours (good or satisfactory results)</td>
</tr>
<tr>
<td>Number of cases when computed position of the trough deviated from the real position more than ± 4° of longitude in 24 hours (unsatisfactory results)</td>
</tr>
</tbody>
</table>
Fig. 1. Results of computations with Rossby's formula.

Double lines enclose errors less than 4° of longitude (measured at latitude 35°).

The population of the sample is too small for reliable statistical conclusions, but a brief examination of 500 mb charts for summer and autumn months of the 1956-57 season shows that Rossby's long waves occur even less frequently in these seasons than in September-October in the Australian region and when they do occur the latitudinal belt most suitable for calculations in most cases is south of latitude 35° where computation of is unreliable.
4. CONCLUSIONS

The results obtained do not imply that Rossby's formula is valueless in itself, they show only that the network in Australia is not sufficient for quantitative computations with this formula.

Apart from Rossby's formula there are other "objective" methods of the computation of the displacement of troughs and ridges on 500 mb - Petterssen's "grid" method, etc. (Staff Members, University of Chicago 1956). Unfortunately all these methods require more accurate 500 mb charts than does Rossby's formula and their application to Australian conditions would be even more difficult.

It may be concluded that, for the time being, there is little opportunity of application of "objective" methods (theoretical formulae) for forecasting 500 mb level charts in Australia. It seems that an improvement of upper level prognostic charts might be achieved by synoptic investigations of the upper level flow in the Australian region. Collection of upper level charts for about 9 years (1949-1957) would be sufficient to begin an investigation with the object of obtaining some empirical rules for forecasting flow patterns typical of the Australian region.

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