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# ON THE VORTICITY THEOREM

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Abstract: It is shown that the Rossby $\beta$ term arises from the vortex tube term in the vertical vorticity equation in spherical - polar co-ordinates. This result, formally due to Karelsky, lends itself to a physical interpretation and a simple graphical representation which are discussed.

In view of the basic importance of the long waves in the westerlies, it is justifiable to explore the origin of the $\beta$ term in the equation of vertical component of relative vorticity. This term appears of course in equations governing less simple motion than long waves, but for simplicity we shall here consider the case of nondivergent horizontal motion in a frictionless barotropic atmosphere in which stable long waves are feasible.

Rossby did not derive the differential equation governing long waves from the general vector equation of the variation of vorticity. Instead he made what amounted to a postulation that the vertical vorticity component referred to an inertial framework of co-ordinates i.e. absolute vertical vorticity, is conserved individually.

It is immaterial for the following considerations that Rossby, although he designated an increment a $d\varphi$, did not mainly use spherical polar co-ordinates. Errors arising from his approximations occurring in the expression for relative vorticity in that co-ordinate system and in the differential operator d/dt are not as a rule large compared with errors in wind gradients deduced from observation. The further assumptions and approximations introduced in equation (3) leading to the solution for long waves in westerlies are quite a different matter.

The question as to whether the $\beta$ term in Rossby's vorticity theorem written in Cartesian co-ordinates, lends itself to an unambiguous physical interpretation was recently raised by Karelsky (1) and discussed further by Clarke and Radok (2). According to the latter authors the physical implications of the spherical shape of the earth in formulating the vorticity equation governing large-scale flow in a simplified equivalent of the atmosphere, were given justice in Rossby's treatment. It seems however as if in the heat of argument a crucial point had been turned into a crux criticorum *.

2. On a later occasion Karelsky (3) has given a formal mathematical derivation to show that the use of Cartesian co-ordinates does not strictly permit of any but trivial solutions for non-divergent large-scale motion in a barotropic atmosphere. While this demonstration is part of the earlier controversy, Karelsky on that occasion also gave a rigorous derivation of the basic vorticity equations in spherical polar co-ordinates. From this the writer was led to the interpretation of the vorticity theorem set out below.

Omitting the divergence $\partial u_i/\partial x_i$ and the solenoid term, i.e. the vector product of $\partial/\partial x_i\ (1/\rho)$ into $\partial p/\partial x_i$ where $\rho$ is density and p pressure, the vorticity equation in a framework of Cartesian co-ordinates rotating with the earth reduces to

$$\frac{d\zeta_i}{dt} - (\zeta_j + 2\Omega_j)\frac{\partial u_i}{\partial x_j} = 0 \qquad (1)$$

or

$$\frac{\partial\zeta_i}{\partial t} + u_j\frac{\partial\zeta_i}{\partial x_j} - (\zeta_j + 2\Omega_j)\frac{\partial u_i}{\partial x_j} = 0 \qquad (1a)$$

u, $\zeta$ and $\Omega$ denoting relative velocity, relative vorticity and earth's angular velocity respectively. Alternatively, equation (1) may be written as

$$\frac{d\eta_i}{dt} = \eta_j\frac{\partial u_i}{\partial x_j} \qquad (1b)$$

where

$\eta_i = \zeta_i + 2\Omega_i$ is the absolute vorticity vector and the identity $d\Omega_i/dt \equiv 0$ has been introduced.

The second term in equation (1) is the vortex tube (twisting) term by which those variations in component vorticities are accounted for that result from gradients of velocity components in the direction

* puzzle for the critics.

of the vorticity components. We are here only interested in the radial (vertical) component of equation (1). In spherical polar co-ordinates r, $\varphi$ , $\lambda$ , the r-component of equation (1a) turns out to be (see Karelsky (3)):

$$\frac{\partial \zeta_r}{\partial t} + w\frac{\partial \zeta_r}{\partial r} + \frac{u}{a\cos\varphi}\frac{\partial \zeta_r}{\partial \lambda} + \frac{v}{a}\frac{\partial \zeta_r}{\partial \varphi} - \frac{v\zeta_\varphi}{a} - \frac{u\zeta_\lambda}{a}$$

$$- \zeta_r\frac{\partial w}{\partial r} - \frac{\zeta_\lambda}{a\cos\varphi}\frac{\partial w}{\partial \lambda} - \frac{\zeta_\varphi}{a}\frac{\partial w}{\partial \varphi} + \frac{\zeta_\varphi v}{a} + \frac{\zeta_\lambda u}{a} \qquad (2)$$

$$- 2\Omega_r\frac{\partial w}{\partial r} \qquad - \frac{2\Omega_\varphi}{a}\frac{\partial w}{\partial \varphi} + \frac{2\Omega_\varphi v}{a} = 0$$

where $u = \cos\varphi\ a\ d\lambda/dt$, $v = a\ d\varphi/dt$ and a is the earth's radius and $\zeta_r$, the relative vertical vorticity, is given by

$$\zeta_r = \frac{1}{a\cos\varphi}\left(\frac{\partial v}{\partial \lambda} - \frac{\partial(\cos\varphi\ u)}{\partial \varphi}\right).$$

It is seen that the first four metric terms * cancel out. As we have assumed horizontal motion all terms containing the vertical velocity w (= dr/dt) may be omitted in our case and equation (2) reduces to

$$\frac{\partial \zeta_r}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial \zeta_r}{\partial \lambda} + \frac{v}{a}\left(\frac{\partial \zeta_r}{\partial \varphi} + 2\Omega\cos\varphi\right) = 0 \qquad (3)$$

since $\Omega_\varphi = \Omega\cos\varphi$ . It is thus readily seen that the $\beta$ term i.e. $2\Omega\cos\varphi\ a^{-1}v \equiv \beta v$ is the metric part of the vertical component of the vortex tube term in equations (1) or (1a).

It will be noted that at no stage of the derivation is there any question of differentiating the Coriolis parameter, i.e. the vertical component of the earth's vorticity $f \equiv 2\Omega_r \equiv 2\Omega\sin\varphi$ . The

* "Metric" terms have to be added for a spherical-polar co-ordinate system to terms in the expansion of d/dt providing in that system the operator d/dt is defined similarly as in Cartesian co-ordinates, namely as

$$\frac{d}{dt} = \frac{\partial}{dt} + \frac{u}{a\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial r}$$

interpretation of the $\beta$ term in the first place is therefore as follows: owing to the constraint of the motion in spherical surfaces, $\zeta_r$ vorticity is created wherever the motion has a meridional component and this generation is equal to a meridional 'angular transport' of $\Omega_\varphi$ vorticity given by $2\Omega_\varphi d\varphi/dt \equiv 2\,\Omega \cos\varphi\ a^{-1}v$.

3. It is of course entirely justified to introduce at this stage the identity $\Omega_\varphi \equiv \partial\Omega_r/\partial\varphi$ which makes it possible to write equation (3) in the form

$$\frac{d\,(\zeta_r + f)}{dt} = 0 \tag{3a}$$

However, it is important to bear in mind that equation (3a) has been derived from the vector equation (1) which states that in non-divergent motion in a frictionless barotropic atmosphere a variation of relative vorticity $\zeta_i$ is caused by 'twisting' of vortex tubes only. In horizontal motion all that remains of the vortex tube term is the $\beta$ term.

To obtain a clearer picture of the meaning of the $\beta$ term, we shall transform the vector $2\Omega_j\,\partial u_i/\partial x_j$. Using now the notation $\underline{A}$ for a vector $A_i$ and $\underline{\nabla}$ for the gradient operator, it follows from a well known transformation (see e.g. Brand (4)):

$$-\ 2\underline{\Omega}\cdot\underline{\nabla}\,\underline{V} = 2\underline{\Omega}\times(\underline{\nabla}\times\underline{V}) = 2\underline{\Omega}\times\underline{\zeta} \tag{4}$$

since $\nabla\Omega = \nabla\times\Omega = \nabla.\Omega \equiv 0$, and $\underline{\nabla}.\underline{V} = 0$ on account of the assumption of non-divergence. The vertical component of (4) is thus given by

$$(2\underline{\Omega}\times\underline{\zeta})_r = -\,2\,\Omega_\varphi\zeta_\lambda = -\,2\Omega_\varphi\left(a^{-1}\frac{\partial\omega}{\partial\varphi} - \frac{\partial v}{\partial r} - va^{-1}\right) \tag{4a}$$

As for horizontal motion in a barotropic atmosphere $w = \partial w/\partial\varphi = \partial v/\partial r = 0$ it follows

$$(2\underline{\Omega}\times\underline{\zeta})_r = 2\Omega_\varphi a^{-1}v \equiv 2\Omega_\varphi\frac{d\varphi}{dt} \tag{4b}$$

Let $\underline{\omega}$ be the angular velocity vector determining the meridional (horizontal) component motion; equation (4b) then may be written as

$$(2\underline{\Omega}\times\underline{\zeta})_r = (2\underline{\Omega}\times\underline{\omega})_r \tag{5}$$

From (2), (4b) and (5) it is readily seen that the $\beta$ term represents generation of $\zeta_r$ vorticity by a twisting of $\Omega_\varphi$ vorticity due to (curvilinear) motion in the meridian of the globe. In Fig. 1 this interpretation is shown graphically.

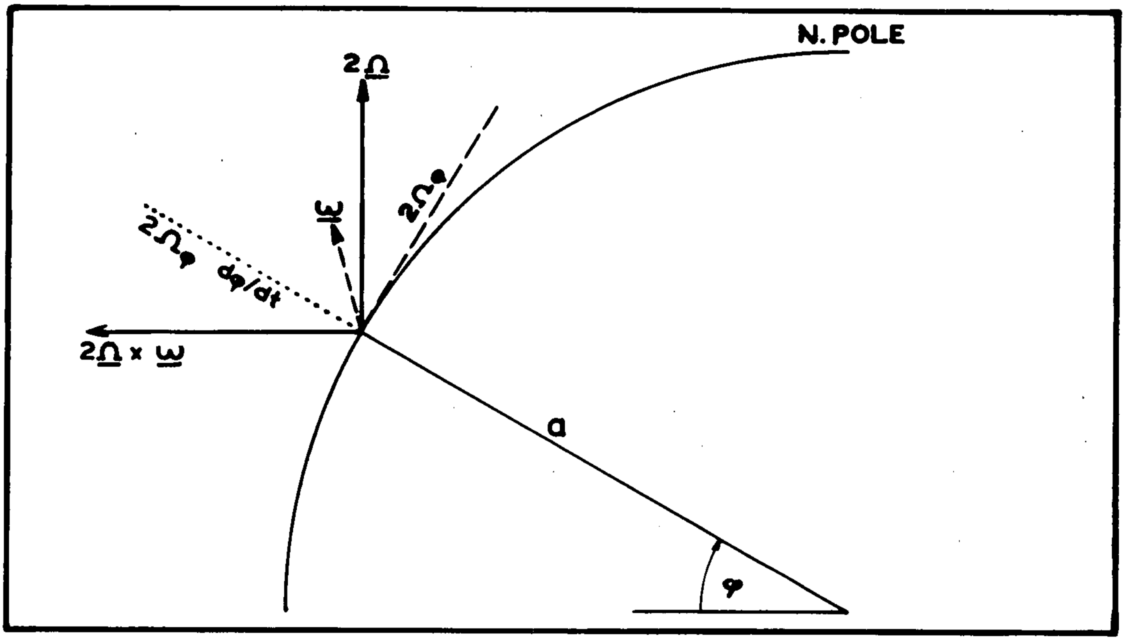


FIG.1 INTERPRETATION OF $\beta$ TERM AS VERTICAL COMPONENT OF A VORTEX TUBE (TWISTING) TERM IN HORIZONTAL MOTION ON A SPHERE. THE VECTOR $\underline{\omega}$ IS PERPENDICULAR TO THE PLANE OF THE PAPER POINTING TOWARDS THE READER.

4 The above deductions have shown that in the case of horizontal motion the conservation of absolute vertical vorticity is not so much a theorem in its own right as the consequence of a more general theorem applicable, as demonstrated by Rossby, to large scale motion.

According to equation (4) this theorem may be written as:

$$\frac{d^{*}\zeta}{dt} = 0 \qquad (6)$$

where the operator $d^{*}/dt$ is defined by:

$$\frac{d^{*}..}{dt} = \frac{d.}{dt} + 2\underline{\Omega} \times \ ...$$

and the operator d/dt is the same as defined in the footnote .

Various interesting conclusions can be drawn from the application of the vorticity equation corresponding to equations (1) or (6) but containing the divergence term. It suffices to point out that Kuo (5) has derived a quasi-nondivergent vorticity equation containing, among others, terms which "originate partly from the divergence term and partly from the twisting term ($\zeta_j \ \partial u_i/\partial x_j$ in equation (1)) and represent an overturning or convective process in a vertical plane perpendicular to the horizontal isotherms". It is thus possible that certain physical processes will be overlooked or misrepresented when the $\beta$ term in equation (3) or in the more generally valid equivalent of this equation, is interpreted as an advective term $va^{-1} \ \partial f/\partial\phi$, as has been done since the days Rossby pointed out its importance and formulated the conservation theorem.

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