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MEANS AND STANDARD DEVIATIONS OF TEMPERATURE PRESSURE AND DENSITY AT CONSTANT HEIGHT

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Abstract: Generally statistics derived from radiosonde observations relate to standard pressure surfaces. Occasionally they are required for particular heights. Equations are given whereby statistics for geopotential heights can be derived from those relating to pressure surfaces.

Limited tests of these equations indicated that they give values close to those obtained from daily values at constant height. Also, where great accuracy is not necessary, the tests suggest that standard deviations of temperature and pressure for particular heights can be interpolated from the standard deviations at constant pressure.

1. INTRODUCTION

Nowadays, statistics derived from radiosonde observations generally relate to standard pressure surfaces. Occasionally there is a requirement for means and standard deviations of temperature, pressure and density at particular heights or geopotentials. This note gives formulae by which these statistics can be derived from statistics relating to constant pressure surfaces. The results of limited tests of these formulae are also discussed.

Strictly, in most cases where temperature, T, appears in the formulae, it should be replaced by the virtual temperature but no great inaccuracies will result by regarding the air as dry particularly above 700 mb.

2. FORMULAE

(i) Standard deviation of temperature at constant geopotential.

Considering the inverse problem to that treated by Crossley (1950) it can be shown that, to a first approximation

$${}_h\sigma_T^2 = {}_p\sigma_T^2 - 2\frac{\partial T}{\partial h}\,{}_pr_{hT}\,{}_p\sigma_T\,{}_p\sigma_h + \left(\frac{\partial T}{\partial h}\right)^2 {}_p\sigma_h^2 \qquad (1)$$

where ${}_h\sigma_T$ and ${}_p\sigma_T$ are the standard deviations of temperature at constant geopotential and pressure respectively, when h and p refer to the same average level, $\partial T/\partial h$ and $(\partial T/\partial h)^2$ are the mean values of $\partial T_p/\partial h$ and $(\partial T_p/\partial h)^2$ respectively; ${}_pr_{hT}$ is the coefficient of correlation between geopotential and the temperature on a constant pressure surface and ${}_p\sigma_{hT}$ is the standard deviation of geopotential at constant pressure.

Stidd (1954) has shown that

$${}_pr_{hT}\;{}_p\sigma_T = \bar{T}\,\frac{d\;{}_p\sigma_h}{d\;h}$$

where $\bar{T}$ is the mean temperature at pressure p. Equation (1) can then be written

$${}_h\sigma_T^2 = {}_p\sigma_T^2 - 2\bar{T}\,\frac{\partial T}{\partial h}\,\frac{d\;{}_p\sigma_h}{d\;h}\;{}_p\sigma_h + \left(\frac{\partial T}{\partial h}\right)^2 {}_p\sigma_h^2 \qquad (2)$$

(ii) Standard deviation of pressure at constant geopotential. Garriock (1954) and Mook (1958) have indicated that the relation

$${}_h\sigma_p = \frac{g\,p}{R\,T}\;{}_p\sigma_h \qquad (3)$$

can be used to obtain the standard deviation of pressure at constant geopotential (${}_h\sigma_p$) in terms of the standard deviation of geopotential at constant pressure (${}_p\sigma_h$) when p and h again refer to the same average level, $\bar{T}$ is the mean temperature of this level, g is the acceleration of gravity and R the gas constant for dry air.

(iii) Mean and standard deviation of density at constant geopotential.

From the equation of state, when the bars indicate mean values at constant geopotential

$$\bar{p} = R\,\overline{\rho T}$$

or

$$p = R\,(\bar{\rho}\,\bar{T} + {}_hr_{\rho T}\;{}_h\sigma_\rho\;{}_h\sigma_T) \qquad (4)$$

where ${}_hr_{\rho T}$ is the coefficient of correlation between density (ρ) and temperature at constant geopotential.

As $|r_{\rho T}| < 1$ and the highest values of ${}_h\sigma_\rho$ and ${}_h\sigma_T$ are of the order $0.03\bar{\rho}$ and $0.02\bar{T}$,

$$|r_{\rho T}\ {}_h\sigma_\rho\ {}_h\sigma_T| < 6 \times 10^{-4}\,\bar{\rho}\,\bar{T}$$

so the second term on the right hand side of equation (4) is negligible compared with the first. A check on equation (4) (see Table 2 below) using means of daily values of pressure, temperature and density at constant geopotentials in a summer and a winter month at one station confirms this, so we can write

$$\bar{p} = R\bar{\rho}\bar{T} \qquad (5)$$

Logarithmic differention of the equations of state and use of equation (5) gives

$$\frac{\delta T}{\bar{T}} = \frac{\delta p}{\bar{p}} - \frac{\delta \rho}{\bar{\rho}} \qquad (6)$$

where δT, δp and $\delta\rho$ can be regarded as departures from the mean values of T, p and ρ at constant geopotentials.

Squaring equation (6) and summing over all departures gives

$$\frac{{}_h\sigma_T^2}{\bar{T}^2} = \frac{{}_h\sigma_p^2}{\bar{p}^2} + \frac{{}_h\sigma_\rho^2}{\bar{\rho}^2} - \frac{2}{\bar{p}\bar{\rho}}\ {}_h r_{p\rho}\ {}_h\sigma_p\ {}_h\sigma_\rho \qquad (7)$$

where ${}_h r_{p\rho}$ is the coefficient of correlation between p and ρ at constant geopotential h.

Now $\quad {}_h\sigma_p^{\,2} = \overline{p^2} - \bar{p}^2$

Differentiating this with respect to geopotential gives

$$\begin{aligned} {}_h\sigma_p \frac{d\,{}_h\sigma_p}{dh} &= \overline{p\frac{dp}{dh}} - \bar{p}\frac{d\bar{p}}{dh} \\ &= -g\,(\overline{p\rho} - \bar{p}\,\bar{\rho}) \\ &= -g\ {}_h r_{p\rho}\ {}_h\sigma_p\ {}_h\sigma_\rho \end{aligned}$$

Substituting in equation (7) gives

$${}_h\sigma_\rho = \bar{\rho}\left(\frac{{}_h\sigma_T^2}{\bar{T}^2} - \frac{{}_h\sigma_p^2}{\bar{p}^2} - \frac{2\,{}_h\sigma_p}{g\bar{\rho}\bar{p}}\frac{d\,{}_h\sigma_p}{dh}\right)^{\frac{1}{2}} \qquad (8)$$

or

$${}_h\sigma_\rho = \bar{\rho}\left(\frac{{}_h\sigma_T^2}{\bar{T}^2} - \frac{{}_h\sigma_p^2}{\bar{p}^2} + \frac{2\,{}_h\sigma_p}{\bar{p}}\frac{d\,{}_h\sigma_p}{d\bar{p}}\right)^{\frac{1}{2}} \qquad (8a)$$

(iv) Standard deviation of density at constant pressure.

On the basis of equation (5) which also holds approximately for a constant pressure surface and equation (6) the standard deviation of density at constant pressure is given approximately by

$$ {}_p\sigma_\rho = \frac{\bar{\rho}}{\bar{T}} \; {}_p\sigma_T \qquad (9) $$

3. TEST OF ACCURACY OF FORMULAE

The formulae were checked against statistics obtained using daily values of pressure temperatures and density at particular geopotentials for the months of February and July 1958 at Woomera.

Mean soundings for February and July 1958 were constructed on the basis of the mean temperature and mean geopotential at standard pressure surfaces. From the pressure-height curve pressures and temperatures at the surface, 2500, 5000, etc. gp ft were read off these mean soundings and the density corresponding to the pressure and temperature at each height was calculated. Of course, provided the standard pressure levels are so located as to show significant changes in lapse rate through the mean sounding, no great differences would be expected between the mean pressures and temperatures for particular heights from daily soundings and the pressures and temperatures obtained from the mean soundings. However, it was necessary to derive these values to check the relation for density.

The results (Tables 1 and 2) show that for pressure the differences were less than 2 mb (as values are given to the nearest millibar) except at 2,500 ft; for temperature the differences were no more than about 1°C; and for density the differences were less than 2 gm m^{-3} except for the 7 gm m^{-3} difference at 2,500 gp ft in July. This large difference and, of course, the differences in pressure and temperature at this height are due to the presence of a mean surface inversion (the soundings were made about 8a.m. local time) whose top in July, 1958, was usually about or just below 930 mb. Values interpolated from those at the surface and 900 mb are therefore subject to errors which, however, could be considerably reduced by using mean data at 950 mb.

Table 3, column (a) gives the values of the standard deviation of temperature at particular heights, as calculated from daily soundings and, in column (b), as calculated from equation (2) using lapse rates from the mean sounding from the month (not reproduced) and values of $d\,{}_p\sigma_h/dh$ from the distribution of ${}_p\sigma_h$ with height as gives in Fig 1. Generally the values given by equation (2) are within 0.3°C of the values calculated from individual soundings.

Table 1 - Woomera pressure and temperature at particular geopotentials for February and July, 1958 (a) means from actual values on individual days and (b) as interpolated from mean soundings based on mean geopotentials and temperatures at constant pressure

Geopotential	Pressure (mb)				Temperature (°C)			
(10^3 gp ft)	Feb.	1958	July	1958	Feb.	1958	July	1958
	a	b	a	b	a	b	a	b
Surface	995	995	996	996	21.2	21.2	9.1	9.1
2.5	928	926	927	930	19.9	19.6	8.9	8.0
5	848	849	844	845	16.4	16.3	4.0	4.1
10	707	707	697	697	8.3	8.3	-4.2	-4.2
15	585	584	572	572	- 1.2	- 1.3	-13.1	-13.1
20	482	483	467	467	-11.2	-16.7	-22.4	-22.4
25	394	395	379	380	-20.9	-21.0	-32.0	-31.3
30	318	319	304	304	-33.3	-32.7	-39.6	-39.7
35	255	255	243	243	-44.3	-43.5	-47.3	-47.3
40	203	203	192	193	-53.8	-54.0	-54.3	-54.3
45	158	159	151	152	-62.3	-61.7	-60.9	-60.5
50	124	124	118	119	-68.3	-67.6	-63.3	-62.5
55	96	96	92	92	-72.1	-71.5	-63.8	-63.0
60	74	74	72	72	-69.9	-68.8	-60.3	-60.2

Table 2 - Woomera density (Kgm m^{-3}) at constant geopotential for February and July, 1958.

(a) the actual mean densities as calculated from the values of pressure and temperature on constant geopotential on individual days;

(b) calculated from the expression $\bar{\bar{P}}/R\bar{\bar{T}}$ where $\bar{P}$ and $\bar{T}$ are the values in column (b) of Table 1.

Geopotential	February 1958		July 1958	
(10^3 gp ft)	a	b	a	b
Surface	1.178	1.178	1.230	1.229
2.5	1.103	1.102	1.145	1.152
5	1.021	1.022	1.061	1.062
10	0.874	0.875	0.902	0.903
15	0.749	0.749	0.766	0.766
20	0.641	0.641	0.648	0.649
25	0.544	0.546	0.547	0.547
30	0.462	0.462	0.454	0.453
35	0.387	0.387	0.374	0.374
40	0.322	0.323	0.306	0.307
45	0.263	0.262	0.248	0.249
50	0.210	0.210	0.196	0.197
55	0.166	0.166	0.152	0.153
60	0.127	0.126	0.117	0.118

Table 3 - Woomera standard deviation of temperature (°C) at constant geopotential ($_h\sigma_T$) for February and July, 1958, (a) actual value as calculated from daily soundings, and (b) calculated from equation (2). Column (c) gives the value of $_p\sigma_T$ for the pressure corresponding to geopotential "h" interpolated from values of $_p\sigma_T$ at standard pressure levels.

Geopotential	February 1958			July 1958		
(10^3 gp ft)	a	b	c	a	b	c
Surface	5.2	5.1	5.2	3.1	3.1	3.1
2.5	5.8	5.5	5.5	3.2	3.5	3.5
5	4.2	4.1	4.1	3.8	3.9	3.9
10	2.2	2.2	2.1	3.0	2.9	2.8
15	3.0	3.0	2.8	3.3	3.3	3.2
20	3.1	2.9	2.7	4.7	4.4	4.2
25	3.3	3.2	2.5	4.5	4.7	4.6
30	3.2	3.0	2.4	4.9	5.0	4.9
35	3.5	3.5	2.9	3.8	4.3	4.0
40	3.7	3.8	3.6	2.9	3.2	3.4
45	2.8	2.7	2.7	4.5	4.7	4.9
50	2.8	2.9	3.1	3.4	3.6	3.8
55	4.4	4.3	4.2	3.3	3.2	3.2
60	3.1	3.2	3.1	2.8	3.2	3.2

Column (c) in Table 3 gives values of $_p\sigma_T$ for the pressures, corresponding to the particular heights, interpolated from values of $_p\sigma_T$ at standard pressure levels. It will be seen that in a few cases, the differences between these values and those in column (b) are 0.3°C or greater, the largest differences (0.8°C) occurring between 23,000 and 35,000 gp ft in February.

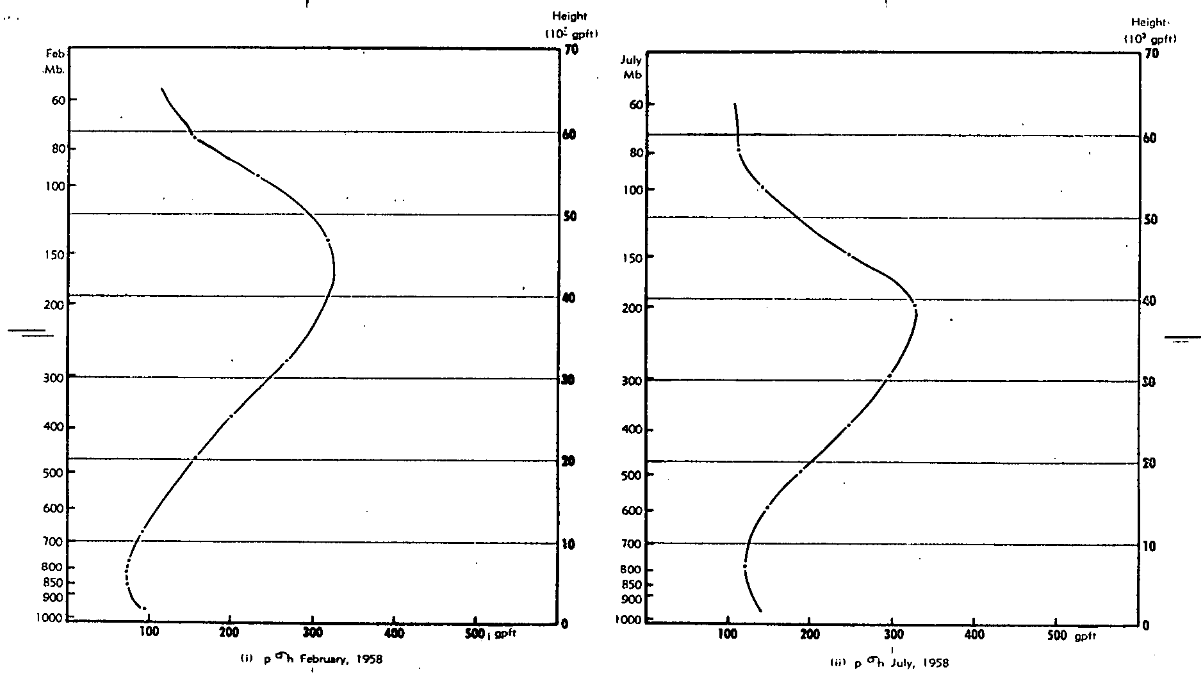

Fig. 1. Woomera standard deviation of geopotential at constant pressure for (i) February, 1958 and (ii) July, 1958.

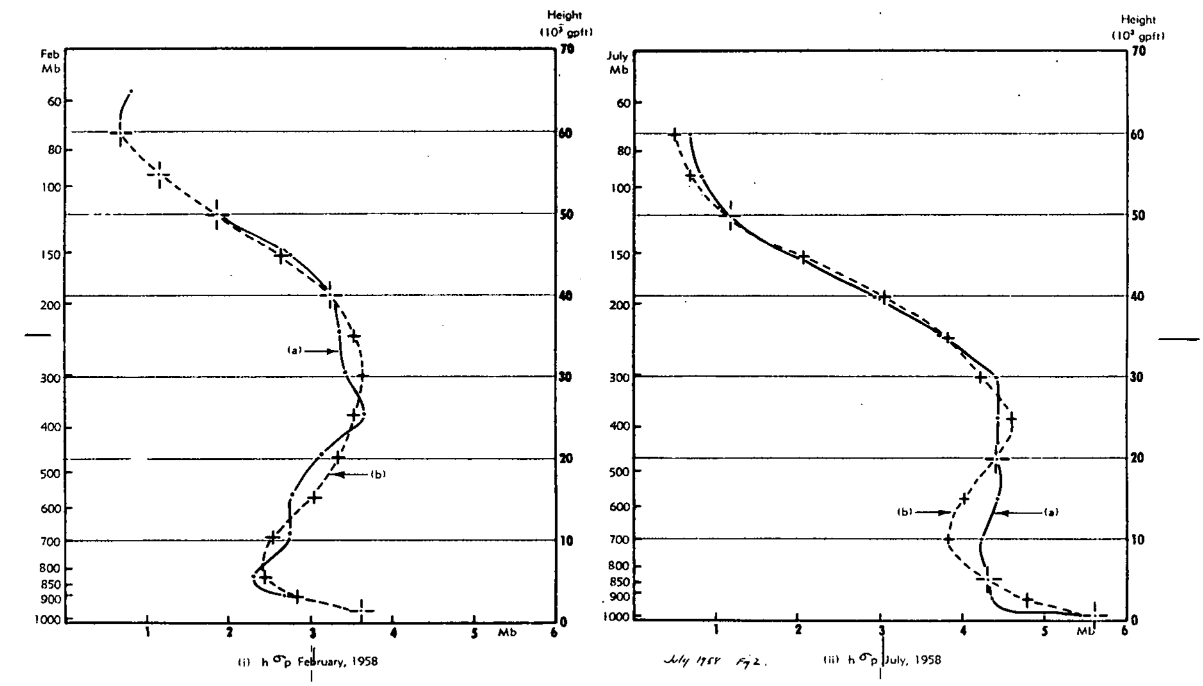

Fig. 2. Woomera standard deviation of pressure at constant geopotential. (i) February, 1958 and (ii) July, 1958. Curves (a) and dots refer to values calculated from daily soundings; curves (b) and crosses to values calculated from equation (3).

Crossley found that at Larkhill and Agra $_h\sigma_T$ was greater than $_p\sigma_T$ by values up to a maximum of about 1°C in the troposphere (where $\partial T/\partial h < 0$) and that $_h\sigma_T \approx {}_p\sigma_T$ in the stratosphere (where $\partial T/\partial h \approx 0$).

A similar situation is seen at Woomera in February, 1958, but in July $_p\sigma_T$ (column (c)) is greater than the calculated value (column (b)) of $_h\sigma_T$ from 40,000 to 50,000 gp ft. In the mean sounding for July $-\partial T / \partial h$ is positive to 53,500 gp ft, negative above. Fig 1 (ii) shows $d\,{}_p\sigma_h/dh$ negative below 6,500 and above 39,000 gp ft. Equation (2) then indicates that, provided the magnitude of the second term is greater than that of the third term on the right hand side (as is the case), $_h\sigma_T$ should be less than $_p\sigma_T$ below 6,500 gp ft and between 39,000 and 53,500 gp ft. Below 6,500 gp ft the values of the second and third terms are so small that $_p\sigma_T$ approximately the same as the calculated value of $_h\sigma_T$ there.

Although interpolated values of $_p\sigma_T$ as an estimate of $_h\sigma_T$ are generally a poorer approximation than values obtained from equation (2) the former values approximate $_h\sigma_T$ to within about 1°C which is probably sufficiently accurate for most practical purposes when the accuracy of temperature measurements is taken into consideration. However if greater accuracy is desirable equation (2) should obviate the necessity to extract data for particular heights from individual soundings when statistics relating to pressure surfaces are available.

Table 4 shows that, in most cases, the differences between the standard deviation of pressure at particular heights calculated from daily soundings and from equation (3) are no greater than 0.3 mb. Fig. 2 shows the distribution of $_h\sigma_p$ with height for both methods of calculation.

Table 4 - Woomera standard deviation of pressure (mb) at constant geopotential for February and July, 1958 (a) actual value as calculated from daily soundings; (b) calculated from equation (3).

Geopotential	February, 1958		July, 1958	
(10^3 gp ft)	a	b	a	b
Surface	3.7	3.6	5.6	5.7
2.5	2.7	2.8	4.3	4.8
5	2.2	2.4	4.3	4.3
10	2.7	2.5	4.2	3.8
15	2.7	3.0	4.4	4.0
20	3.1	3.3	4.4	4.4
25	3.6	3.5	4.4	4.6
30	3.4	3.6	4.4	4.2
35	3.3	3.5	3.7	3.8
40	3.2	3.2	2.9	3.0
45	2.7	2.6	1.9	2.0
50	1.8	1.8	1.1	1.1
55	1.1	1.1	0.7	0.6
60	0.6	0.6	0.6	0.4

Table 5, which has been included for later reference in connection with the standard deviation of density at constant height, shows that the standard deviation of density at constant pressure as calculated from equation (9) differs from that calculated from daily soundings by no more than 1 gm m^{-3}.

Table 5 - Woomera standard deviation of density (kgm m^{-3}) at constant pressure for February and July, 1958 (a) actual value as calculated from daily soundings; (b) calculated from equation (9).

Pressure	February, 1958		July, 1958	
(mb)	a	b	a	b
900	.0206	.0213	.0146	.0148
850	.0145	.0149	.0146	.0132
800	.0092	.0091	.0131	.0129
700	.0065	.0067	.0102	.0094
600	.0080	.0079	.0085	.0091
500	.0067	.0068	.0104	.0111
400	.0057	.0057	.0097	.0106
300	.0045	.0044	.0095	.0102
200	.0053	.0054	.0046	.0046
150	.0030	.0028	.0056	.0057
100	.0038	.0038	.0027	.0025
80	.0024	.0024	.0021	.0027
60	.0009	.0010	.0012	.0012

Table 6, column (a) gives values of the standard deviation of density at particular heights as calculated from daily sounding and column (b) gives values as calculated from equation (8), using previous calculations of $\bar{p}$, $\bar{T}$, $\bar{\rho}$, ${}_h\sigma_p$ and ${}_h\sigma_T$ given in columns (b) of earlier tables and values of $d\,{}_h\sigma_p / dh$ obtained from curves (b) of Fig 2. Except near the surface the differences generally are less than 2 gm m^{-3} (the values in Table 6 are given to the nearer gm m^{-3}). Column (c) gives values of the standard deviation at constant pressure interpolated, for the mean pressures corresponding to particular heights, from the values obtained in column (b) of Table 5. The differences between these values and those of column (a) are generally no more than 3 gm m^{-3}. Differences in temperatures of up to 1°C are equivalent, at constant pressure, to differences of up to 4 gm m^{-3} in density in the atmospheric range of temperatures.

Table 6 - Woomera standard deviation of density (kgm m^{-3}) at constant geopotential for February and July, 1958 (a) actual value as calculated from daily soundings; (b) calculated from equation (8). Column (c) gives the standard deviation of density at constant pressure interpolated from values given in column (b) of Table 5 for the mean pressure corresponding to the particular geopotential.

Geopotential	February, 1958			July, 1958		
(10^3 gp ft)	a	b	c	a	b	c
Surface	.025	.024	-	.018	.021	-
2.5	.024	.022	.022	.016	.016	.015
5	.015	.015	.015	.016	.015	.013
10	.006	.005	.007	.009	.008	.009
15	.006	.006	.008	.008	.007	.009
20	.005	.004	.007	.008	.008	.011
25	.003	.004	.006	.008	.009	.011
30	.003	.002	.005	.009	.009	.010
35	.006	.004	.005	.008	.008	.008
40	.006	.006	.005	.009	.008	.005
45	.005	.005	.004	.008	.008	.005
50	.005	.006	.003	.005	.005	.004
55	.005	.005	.004	.003	.003	.003
60	.003	.003	.002	.002	.002	.002

If errors of 1°C can be tolerated in the standard deviation of temperature, standard deviations of density obtained in column (c) would be sufficiently accurate. However, if it is desirable that equation (2) be used for the standard deviation of temperature, equation (8) should be used for the standard deviation of density. Incidentally equation (8) is much simpler than Mook's (1958) equation for the standard deviation of density at constant height.

4. CONCLUSION

These limited tests suggest that monthly means of density and, incidentally, pressure and temperature at constant height can be obtained with sufficient accuracy from soundings constructed from the mean geopotentials and temperatures at standard pressures except just above the surface when strong surface inversions are frequently present.

The standard deviation of pressure at constant height is given with sufficient accuracy by equation (3) while the standard deviations of temperature and density can be obtained by interpolation from the standard deviations at standard pressures or, if greater accuracy is required, by the use of equations (2) and (8) respectively.

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