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NOTES ON METHODS OF FITTING AN INCOMPLETE GAMMA DISTRIBUTION TO A SET OF OBSERVED VALUES

by J.V. Maher

Central Office, Bureau of Meteorology, Melbourne

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Abstract: The paper demonstrates, in detail, methods of fitting an incomplete gamma distribution to a set of observed values using the maximum likelihood method to obtain estimates of the two parameters and thence estimates of values of the phenomenon which will be equalled or exceeded with specified relative frequencies (1 in 100, 1 in 1000).

Calculation of the standard errors and confidence limits of the latter is also demonstrated.

Moran's (1957) example of the Jingellic Flood Flows is re-worked in more detail and the methods are also applied to 63 years of maximum annual 24 hour (9am to 9am) rainfalls at Maragle, N.S.W, the latter results being compared with those obtained by fitting a log normal distribution.

1. THE INCOMPLETE GAMMA DISTRIBUTION

The problem is that we have a set of observations $x_1, x_2 \ldots x_n$ of some natural phenomenon and we suppose that the particular phenomenon x, has a probability distribution f (x) such that the probability that f (x) lies in the interval (x, x + dx) is f (x) dx.

It follows that the probability that any x will be greater than x_0 say, will be

$$P = \int_{x_0}^{\infty} f(x)\, dx \qquad (1)$$

Various functions, log x, log log x and so on have been used for f (x) but Thom (1958) claims that the incomplete gamma function, commonly called the gamma function, may be fitted satisfactorily to various climatological variates.

The gamma distribution is a 2 parameter frequency distribution given by the equation

$$f(x) = \frac{1}{a\Gamma(k)} \cdot e^{-xa^{-1}} (xa^{-1})^{k-1} \qquad \text{(2) (Moran's Notation)}.$$

where $\Gamma(k) = \int_0^{\infty} e^{-x} . x^{k-1} . dx$ is the Gamma function.

(If k is a positive integer $\Gamma(k) = (k-1)!$)

In Thom's notation the distribution is

$$f(x) = \frac{1}{\beta^{\gamma}\Gamma(\gamma)} . x^{\gamma-1} . e^{-x/\beta} \qquad ; \; \beta > 0, \; \gamma > 0 \quad (3)$$

(3) may be rewritten

$$f(x) = \frac{1}{\beta\Gamma(\gamma)} . e^{-x\beta^{-1}} . \frac{x^{\gamma-1}}{\beta^{\gamma-1}}$$

$$= \frac{1}{\beta\Gamma(\gamma)} . e^{-x\beta^{-1}} (x\beta^{-1})^{\gamma-1}$$

(Thus Moran used k for γ and a for β, probably for simplicity in printing and this notation has been followed throughout this paper).

Here x is the random variable, a is a scale parameter and k the shape parameter. The function f (x) is equal to zero for x equals zero and the distribution thus has a zero lower limit for positive x values and is unlimited on the right. It is positively skewed (the mean is greater than the mode), the amount of skew depending inversely on the shape factor k.

It can be shown that the skewness approaches zero with increasing k, (thus showing that the gamma distribution becomes symmetrical for large ( ) and approaches normality slowly as k increases. For k 100 it is approximately normal for climatological applications.

Values of the integral of (2) have been tabulated by Pearson 1951, i.e. values of the probability that any value of x will be less than the tabulated value (expressed as a u value).

The arguments of Pearson's table are u and p, where $u = \frac{x}{a/\sqrt{k}}$ (4)

and $p = k-1$ (5)

The main problem is to estimate the parameters a and k from the sample observations.

This may be done most efficiently by Fisher's 1941 method of maximum likelihood. This consists of maximising what Fisher called the likelihood, or the product of the frequency functions of a sample.

If f (x, a, k) is any frequency function, the likelihood is defined to be

$M = \prod_{i=1}^{n} f(x_i, a, k)$, where $\prod$ indicates multiplication of all terms and x_i is the ith value in a sample of n.

To maximise this expression it is simplest to take logarithms before differentiating and setting to zero.

This gives

$$L = \sum_{i=1}^{n} \ln f(x_i, a, k) \qquad (6)$$

Differentiating partially with respect to a and k gives the maximum likelihood (M.L) differential equations

$$\left.\begin{aligned} \frac{\partial L}{\partial a} &= 0 \\ \frac{\partial L}{\partial k} &= 0 \end{aligned}\right) \qquad (7)$$

Solving these equations gives the M.L. estimates $\hat{a}$ and $\hat{k}$.

Applying (6) to the gamma distribution equation (2) gives

$$L = -nk \ln a - n \ln \Gamma(k) + (k-1) \Sigma \ln x_i - a^{-1} \Sigma x_i \qquad (8)$$

and differentiating as in (7) we get

$$\frac{\partial L}{\partial k} = -n \Gamma'(\hat{k}) \Gamma(\hat{k})^{-1} + \Sigma \ln x_i - n \ln \hat{a} = 0 \qquad (9)$$

where $\Gamma'(k) = \frac{\partial \Gamma}{\partial k}$, and

$$\frac{\partial L}{\partial a} = \hat{a}^{-2} \Sigma x_i - n \hat{k} \hat{a}^{-1} = 0 \qquad (10)$$

where $\hat{a}$ and $\hat{k}$ signify estimates of the true values of a and k respectively.

From (10) we get:

$$\bar{x} = \hat{k} \hat{a} \qquad (11)$$

The term $\Gamma'(\hat{k}) \Gamma(\hat{k})^{-1}$ is called the digamma function $\psi(k)$, values of which can be found in Davis (1933).

Solving equations (9) and (10) directly for values of $\hat{a}$ and $\hat{k}$ is difficult. Moran advocates finding approximate values for a and k and using a graphical method. Thom, on the other hand has developed a direct approximation for $\hat{k}$.

2. APPLICATION TO JINGELLIC DATA

Moran obtains approximate values of (k and a) using the method of moments. This gives

$$\hat{k} = \bar{x}^2 s^{-2} \qquad (12)$$

and $$\hat{a} = s^2 \bar{x}^{-1} \qquad (13)$$

where $$\bar{x} = n^{-1} \Sigma x_i \qquad (14)$$

and $$s^2 = \left\{ \frac{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}}{n-1} \right\} \qquad (15)$$

which from the data (Jingellic river flows) gives $\bar{x} = 393.34$, $s^2 = 36968$, $\hat{k} = 4.178$ and $\hat{a} = 94.066$.

Substituting the calculated and known values of n, $\bar{x}$, $\Sigma \ln x_i$ in equations (9) and (11) we get

$$-50\, \Gamma'(\hat{k}) \Gamma(\hat{k})^{-1} + 293.16 - 50 \ln \hat{a} = 0 \qquad (16)$$

$$393.34 = \hat{k} \hat{a} \qquad (17)$$

Dividing (16) by 50 gives

$$- \Gamma'(\hat{k})\,\Gamma(\hat{k})^{-1} + \frac{293.16}{50} = \ln \hat{a}$$

$$\text{or } -\psi(\hat{k}) + 5.86312 = \ln \hat{a} \qquad (18)$$

Choosing a series of values for k around 4.18 the approximate value, say 3.8, 4.0, 4.2, 4.4, 4.6, 4.8, 5.0, we calculate the corresponding values of a from (18) and interpolate on k until (17) is satisfied. This may be done by drawing a graph of k against ak and reading off the value for $\hat{k}$ which results in a value of 393.34 for ak.

From Table 1 it can be seen that this value is approximately 4.6, which corresponds to a value of 85.59 for $\hat{a}$.

Graphing the values of $\hat{k}$ against $\hat{a}\hat{k}$ and interpolating a value of $\hat{k}$ for $\hat{a}\hat{k}$ = 393.34 gives k = 4.64 and a corresponding value of 84.77 for $\hat{a}$. However, in order to carry through the check of Morans paper the values $\hat{k}$ = 4.60 and $\hat{a}$ = 85.64 have been used.

Pearson's Tables of the Incomplete Gamma Function give values of I (u, p) for various values of u and p where

$$I(u, p) = \frac{1}{\Gamma(p+1)} \cdot \int_0^{v} e^{-v} v^p \, dv$$

where $v = xa^{-1}$, $p = k-1$ and $v = \sqrt{p+1}.u = \sqrt{k}.u$

Thus having found a value of u_o to make I (u, p) = say 0.99 and 0.999 the corresponding x value is obtained from $x = \hat{a}v = \hat{a}\sqrt{\hat{k}}.u_o$

Here the corresponding values are (for $\hat{k}$ = 4.6 and thus for p = 3.6 - see pages 19 and 21 of Pearson),

for I (u, p) = 0.99; $u_o = 5.13$; $v_o = u_o\sqrt{4.6} = 5.13 \times 2.145 = 10.990$

and
for I (u, p) = 0.999, $u_o = 6.58$; $v_o = u_o\sqrt{4.6} = 6.58 \times 2.145 = 14.113$

$$x_{.99} = \hat{a}v_o = 85.64 \times 10.990 = 941.18 = 941.$$

$$x_{.999} = \hat{a}v_o = 85.64 \times 14.113 = 1208.64 \doteq 1209.$$

We now have that 941 thousand acre feet or more has a relative frequency of once in one hundred years and 1209 or more, a relative frequency of once in one thousand years.

Before evaluating these 100 year and 1000 year floods, a goodness of fit test of the distribution, using the parameters estimated, should be carried out.

TABLE 1

$\hat{k}$	$\psi(\hat{k})$	$5.86312-\psi(\hat{k}) = \ln \hat{a}$	$\text{Log}_{10}(3)$ $\log_{10}(\ln \hat{a})$	(4)-.36221 $\text{Log}_{10}(\log_{10}\hat{a})$	Antilog (5) $=\log_{10}\hat{a}$	Antilog (6) $\hat{a}$	log $\hat{k}$	log $\hat{a}\hat{k}$ (6) + (8)	Antilog (9) $\hat{a}\hat{k}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
3.8	1.1977	4.6654	.66886	.30665	2.0262	106.17	.57978	2.60578	403.45
4.0	1.2561	4.6070	.66341	.30120	2.0008	100.19	.60206	2.60286	400.76
4.2	1.3113	4.5518	.65820	.29599	1.9770	94.842	.62325	2.60025	398.34
4.4	1.3637	4.4994	.65313	.29092	1.9539	89.930	.64345	2.59735	395.69
4.6	1.4134	4.4497	.64836	.28610	1.9324	85.586	.66276	2.59516	393.69
4.8	1.4609	4.4022	.64365	.28144	1.9117	81.621	.68124	2.59294	391.78
5.0	1.5061	4.3570	.63919	.27698	1.8923	78.037	.69897	2.59127	390.18

Using logarithms Table 2 was obtained, taking the same x intervals for the χ^2 test as used by Moran (1957).

Items in column (6) must be interpolated from Pearson's tables. Graphical methods appear to be satisfactory, although complex systems of interpolation are given in the explanatory notes. Thus column (8) gives the expected frequency, and the observed frequencies are given in column (9). Details of the χ^2 test are not shown here but the value obtained is not significant, which signifies that the data do not differ significantly from the incomplete gamma distribution. Although the author is aware of the method suggested by Watson (1957) for the χ^2 Goodness of Fit Test, this method has not been used because of the limitations imposed by the data.

Next, the standard errors of the flood estimates must be estimated.

If x_o is a flood estimate from estimated values k and a, then,

$$\delta x_o = \frac{\partial x_o}{\partial k} \cdot \delta k + \frac{\partial x_o}{\partial a} \cdot \delta a \qquad (21)$$

Summing both sides we get

$$\Sigma(\delta x_o) = \frac{\partial x_o}{\partial k} \Sigma(\delta k) + \frac{\partial x_o}{\partial a} \cdot \Sigma(\delta a) \qquad (22)$$

So that

$$x_o = \frac{\partial x_o}{\partial k} \cdot k + \frac{\partial x_o}{\partial a} \cdot a \qquad (23)$$

$$\therefore \left\{ \text{S.E. } (x_o) \right\}^2 = \left(\frac{\partial x_o}{\partial k} \right)^2 \text{Var } (\hat{k}) + 2 \left(\frac{\partial x_o}{\partial k} \right) \left(\frac{\partial x_o}{\partial a} \right) \text{Cov } (\hat{k}, \hat{a}) + \left(\frac{\partial x_o}{\partial a} \right)^2 \text{Var } (\hat{a}) \qquad (24)$$

To calculate $\frac{\partial x_o}{\partial k}$ we evaluate x_o for 2 values of k on either side of $\hat{k}$ and estimate the differential coefficient from the difference.

Here the values of $x_{o_{.99}}$ and $x_{o_{.999}}$ for $\hat{k}$ = 3.6 and 5.6 have been calculated so that the change in x_o for unit change in k gives the estimate of $\frac{\partial x_o}{\partial k}$ (see Table 3).

TABLE 2. Calculation of Expeded Values.

x value $=au\sqrt{k}$ (1)	log $au\sqrt{k}$ (2)	log $a\sqrt{k}$ (3)	log u (2)-(3) (4)	u Antilog (4) (5)	Cumulated Probabilities (Interpolated from Pearson) (6)	Probabilities in ranges (7)	No. in ranges (7) x 50 (8)	Obs. No. (9)
80.2	1.90417	2.26405	$\bar{1}$.64012	.43664	.0057	.0276	1.38	3
131.2	2.11793	2.26405	$\bar{1}$.85388	.71431	.0333	.1215	6.07	5
214.9	2.33221	2.26405	.06816	1.1699	.1548			
351.9	2.54641	2.26405	.28236	1.9159	.4680	.3132	15.66	15
576.3	2.76065	2.26405	.49660	3.1376	.8463	.3783	18.91	20
943.5	2.97474	2.26405	.71069	5.1369	.9902	.1439	7.19	6
>943.6	–	–	–	–	–	.0098	.49	1

Table 3. Calculation of $\frac{\partial x_o}{\partial k}$, $(x_o = u_o \hat{a} \sqrt{\hat{k}})$

Period	$\hat{k}$	p	$\sqrt{\hat{k}}$	$\hat{a}$	u_o	x_o	Δx_o	Mean $\frac{\partial x_o}{\partial k}$
100 yr	3.6	2.6	1.897	85.64	4.96	806.0		
100 yr	4.6	3.6	2.145	85.64	5.13	941.2	135.2	133
100 yr	5.6	4.6	2.366	85.64	5.29	1072.1	130.9	
1000 yr	3.6	2.6	1.897	85.64	6.50	1056.2		
1000 yr	4.6	3.6	2.145	85.64	6.58	1208.6	152.4	149
1000 yr	5.6	4.6	2.366	85.64	6.68	1353.8	145.2	

Thus $\frac{\partial x_o}{\partial k}_{.99} = 133$ and $\frac{\partial x_o}{\partial k}_{.999} = 149$

How $\frac{\partial x_o}{\partial \hat{a}} = v_o$, since $x_o \hat{a}^{-1} = v_o$

To obtain the variances and covariance of $\hat{k}$ and $\hat{a}$ we differentiate (9) and (10) to get

$$\frac{\partial^2 L}{\partial k^2} = -n \left\{ \Gamma(k) \quad \Gamma''(k) - \quad \Gamma'(k)^2 \right\} \Gamma(k)^{-2} = -n \, \psi'(k) \tag{25}$$

($\psi'(k)$, the 1st derivative of $\psi(k)$, is called the trigamma function and is tabulated in Davis (1933) Vol. II).

For $k = 4.6$, $\psi'(k) = .24271$

$$\frac{\partial L^2}{\partial k \partial a} = -na^{-1} \tag{26}$$

$$\frac{\partial^2 L}{\partial a^2} = -nka^{-2} \tag{27}$$

Now, it can be shown that the variance - covariance matrix

$$\begin{pmatrix} Var(\hat{k}) & Cov(\hat{k}, \hat{a}) \\ Cov(\hat{k}, \hat{a}) & Var(\hat{a}) \end{pmatrix}$$ is the inverse of

the matrix $$\begin{pmatrix} -\frac{\partial^2 L}{\partial k^2} & -\frac{\partial^2 L}{\partial k \partial a} \\ -\frac{\partial^2 L}{\partial k \partial a} & -\frac{\partial^2 L}{\partial a^2} \end{pmatrix}$$

Substituting the values from equations (25), (26) and (27) for $\frac{\partial^2 L}{\partial k^2}$, $\frac{\partial^2 L}{\partial k \partial a}$ and $\frac{\partial^2 L}{\partial a^2}$ and inverting the matrix we obtain the following -

$$Var(\hat{a}) = \frac{\hat{a}^2 \psi'(\hat{k})}{n[\hat{k}\psi'(\hat{k}) - 1]} \qquad (28)$$

$$Var(\hat{k}) = \frac{\hat{k}}{n[\hat{k}\psi'(\hat{k}) - 1]} \qquad (29)$$

$$Cov(\hat{a}, \hat{k}) = \frac{-\hat{a}}{n[\hat{k}\psi'(\hat{k}) - 1]} \qquad (30)$$

Using the values of $\hat{a} = 85.64$, $\psi'(4.6) = .24271$ and $n = 50$ we get the values in Table 4.

TABLE 4.

	$\frac{\delta x_o}{\delta k}$	$\frac{\delta x_o}{\delta a}$	Var $(\hat{a})$	Var $(\hat{k})$	Cov $(\hat{a}, \hat{k})$
$x_{o.99}$	133	10.990	305	.79	-14.71
$x_{o.999}$	149	14.113			

Substituting these values in equation (24) we get,

S.E. $(x_{o.99}) = 89$ and S.E. $(x_{o.999}) = 129$

We then have 95% confidence limits for $x_{o.99}$

to be 941 ± 1.96 x 89

= 941 ± 174 = 1115 and 767 thousand acre feet.

Similarly for $x_{0.999}$ we get

1209 ± 1.96 x 129 = 1461 and 957 thousand acre feet.

These were the results obtained by Moran (1957). In the following section, to provide a worked example the method is applied to the maximum 24 hour rainfalls at Maragle, N.S.W, over a 63 year period.

The once in one thousand year falls and confidence limits are included for purposes of comparison with those calculated by fitting a log normal distribution.

3. APPLICATION OF THE METHOD TO MAXIMUM ANNUAL 24 HOUR (9AM-9AM) RAINFALL TOTALS AT MARAGLE, N.S.W, PERIOD 63 YEARS.

The basic data consist of the 63 max. annual rainfall totals x_i. Rainfall in points.

Thus n = 63. Now $\Sigma x_i = 12465, \Sigma \log_{10} x_i = 143.581$,

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{12465}{63} = 197.86$$

$$\frac{\Sigma \log_{10} x_i}{n} = \frac{143.581}{63} = 2.279 \text{ and } \frac{\Sigma \ln x_i}{n} = 2.279 \times 2.30258 = 5.24758$$

$$\ln \bar{x} = \log_{10} \bar{x} \times 2.30258 = 2.29636 \times 2.30258 = 5.28756$$

Using Thom's method we have

$$A = 5.28756 - 5.24758 = .03998 = .0400$$

$$\hat{k} = \frac{1 + \sqrt{1 + \frac{4 \times .04}{3}}}{4 \times .04} = 12.66$$

$$\hat{a} = \frac{\bar{x}}{\hat{k}} = \frac{197.86}{12.66} = 15.62$$

These values should satisfy the equation

$$- \psi(\hat{k}) + \sum \frac{\ln x_i}{n} = \ln \hat{a}$$

that is $- \psi(\hat{k}) + \quad 5.24758 = \ln \hat{a}$

Taking a series of values for k near $\hat{k}$ and evaluating $\hat{a}\hat{k}$, we get (using logarithms) the values in Table 5.

TABLE 5.

$\hat{k}$	$\psi(\hat{k})$	$\ln \hat{a}$ 5.24758–(2)	$\text{Log}_{10}\hat{a}$ (3) ÷ 2.30258	$\hat{a}$ Antilog (4)	$\hat{a}\hat{k}$ (1) x (5)
(1)	(2)	(3)	(4)	(5)	(6)
12.6	2.49349	2.75409	1.19609	15.708	197.92
12.64	2.49679	2.75079	1.19466	15.655	197.88
12.68	2.50008	2.74750	1.19323	15.604	197.86

Thus $\hat{k} = 12.68$, $\hat{a} = 15.604$

$\hat{a}\sqrt{\hat{k}} = 55.5627$

Having evaluated the parameters it is necessary to use them to obtain expected frequencies using the theoretical distribution and perform a χ^2 test of goodness of fit as shown in Table 6.

$\chi^2_{7-2-1} = 4.3$ is not significant at the 5% level.
$= 4$df

Here again the technique suggested by Watson (1957) for χ^2 tests of significance has not been used for reasons stated previously. From Pearsons tables $x_{o_{.99}} = \hat{a}\sqrt{\hat{k}}.u_{o_{.99}} = 55.56 \times 6.29 = 349.$

$x_{o_{.999}} = \hat{a}\sqrt{\hat{k}}.u_{o_{.999}} = 55.56 \times 7.466 = 415.$

Calculation of $\frac{\partial x_o}{\partial k_{.99}}$ and $\frac{\partial x_o}{\partial k_{.999}}$ are shown in Table 7.

TABLE 6. Calculations for χ^2 test of goodness of fit.

x value ($\hat{a}u\sqrt{k}$)	u value ($\frac{x}{a\sqrt{k}}$)	Cumulated % interpolated from Pearson	% in ranges	Range	Expected number in range f_E	Observed number in range f_o	$f_o - f_E$	$(f_o - f_E)^2$	$\frac{(f_o - f_E)^2}{f_E}$
0	0	0							
75	1.35	.20	.20	0-75	.13)	0)			
99	1.78	1.72	1.52	75-99	.96) 11.78	0) 12	.22	.04	.2
124	2.23	7.21	5.49	100-124	3.46)	4)			
149	2.68	18.68	11.47	125-149	7.23)	8)			
174	3.13	35.33	16.65	150-174	10.49	13	2.51	6.30	.6
199	3.58	54.64	19.31	175-199	12.17	11	1.17	1.37	.1
224	4.03	69.68	15.04	200-224	9.48	12	2.52	6.35	.7
249	4.48	81.89	12.21	225-249	7.69	4	3.69	13.62	1.7
274	4.93	91.36	9.47	250-274	5.97	4	1.97	3.88	.6
299	5.38	94.88	3.52	275-299	2.22)	3)			
324	5.83	97.92	3.04	300-324	1.92) 5.45	2) 7	1.55	2.40	.4
350	6.30	99.02	1.10	325-350	.69)	1)			
>350			.98	>350	.62)	1)		Total	4.3

TABLE 7. Calculation of $\frac{\partial x_o}{\partial k}$, $(x_o = u_o x \hat{a} \times \sqrt{\hat{k}})$

Period	$\hat{k}$	p	$\sqrt{\hat{k}}$	$\hat{a}$	$\hat{a}\sqrt{\hat{k}}$	u_o	x_o	Δx_o	mean $\frac{\partial x_o}{\partial k}$
100 yr	11.6	10.6	3.406	15.6	53.134	6.15	326.77	21.02	
100 yr	12.6	11.6	3.550	15.6	66.380	6.28	347.79	20.42	20.72
100 yr	13.6	12.6	3.688	15.6	57.533	6.40	368.21		
1000 yr	11.6	10.6	3.406	15.6	53.134	7.35	390.53	22.05	
1000 yr	12.6	11.6	3.550	15.6	55,380	7.45	412.58	22.37	22.21
1000 yr	13.6	12.6	3.688	15.6	57.533	7.56	434.95		

Applying now equation (24) we get

$$[S.E.(x_{o_{.99}})]^2 = [(20.72)^2 \times 4.97] + [2 \times 20.72 \times 22.4 \times -6.11] + [(22.4)^2 \times 7.82]$$

$$= 2133.7 - 5671.6 + 3923.8$$
$$= 385.9$$

$$S.E.(x_{o_{.99}}) = (385.9)^{\frac{1}{2}} = 19.65$$

and

$$[S.E.(x_{o_{.999}})]^2 = [(22.21)^2 \times 4.97] + [2 \times 22.21 \times 26.60 \times -6.11] + [(26.60)^2 \times 7.82]$$

$$= 2451.6 - 7219.4 + 5533$$
$$= 765.2$$

$$S.E.(x_{o_{.999}}) = (765.2)^{\frac{1}{2}} = 27.66$$

The 95% confidence limits are therefore

for $x_{o_{.99}}$, $349 \pm 1.96 \times 19.65 = 349 \pm 39 = 310$ and 388

for $x_{o_{.999}}$, $415 \pm 1.96 \times 27.66 = 415 \pm 54 = 361$ and 469

Using the log normal distribution, the Hydrometeorological Section of the Bureau of Meteorology obtained 95% confidence limits as follows:

for $x_{0.99}$, 319 and 417 points

for $x_{0.999}$, 383 and 533 points.

These are considerably wider limits than those obtained using the incomplete gamma distribution

4. CONCLUSIONS

It has been shown that the incomplete gamma distribution may be readily utilised to fit a set of observed values of maximum annual flood flows or 24 hour rainfalls.

Thom's (1958) statement that the incomplete gamma distribution may be fitted satisfactorily to various climatological variates has been supported at least by the example given.

It is clear from the comparison of results for Maragle that the incomplete gamma distribution gives smaller values for the relatively less frequent events than the log-normal distribution. In other words the ogive obtained using the incomplete gamma distribution although unbounded, asymptotically approaches the unit probability ordinate more rapdily than that obtained using the log-normal distribution.

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