AN ANALYSIS OF TWO OCCURRENCES OF LEE WAVES

By J.R. Colquhoun

Central Office, Bureau of Meteorology

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ABSTRACT

Two occurrences of lee waves are analysed. Wavelengths are estimated from radiosonde flights in the lee of the mountain range and compared with theoretical values, calculated by the methods of Foldvik and Caswell applied to radiosonde traces from the windward side of the range. The simplicity of Caswell's method makes it preferable to Foldvik's although they give similar wavelength values.

Maximum vertical velocities and their heights are calculated using both theoretical methods.

1. INTRODUCTION

On two U.S. Airforce "Project Hicat" flights clear air turbulence (CAT) was encountered to the east of the Great Dividing Range of northern New South Wales and southern Queensland. On 28 July 1966 (local time) extensive CAT with vertical accelerations of + .35 g was encountered at 60,500 ft over Brisbane (27° 26' S, 153° 05' E). The next day (29th) severe CAT, with maximum vertical accelerations of + .5 g to - .6 g was encountered at 59,000 ft along a track extending 130 n mi south from Amberley (27° 38' S, 152° 43' E).

These occurrences were to the lee of the range where on both days, although satellite photograph and routine surface observations indicated cloudless conditions, commercial airline pilots reported extensive standing wave activity. The presence of a radiosonde station at Moree (29° 28' S, 149° 51' E, elevation 696 ft) west of the Great Dividing Range, gives the opportunity for a Lyra-Scorer type analysis of the airstream on these days to determine whether standing wave activity could have been forecast.

2. GENERAL BACKGROUND THEORY

Palm and Folvik (1960) assume that the motion is independent of time, with the flow taking place in an x - z plane, with the x axis horizontal and normal to the barrier and the z axis positive upwards; z = 0 corresponds to the ground (in the absence of the barrier); the equations and boundary conditions are linearized, and the density in the basic flow is assumed to be exponential in type, and described by the function

\[ \hat{\rho} = \hat{\rho}(0) e^{-\beta z} \]

Where \( \hat{\rho} \) is the density in undisturbed motion, and

\( \beta \) is a positive constant.
The equation governing the motion is

\[ \nabla^2 \omega + \zeta^2 \omega = 0 \quad \ldots (1) \]

where \( \omega = \frac{w}{e^{\frac{z}{2}}} \) and

\[ \zeta^2 = \frac{S}{U^2} - \frac{1}{U} \frac{dU}{dz} \]

(the Lyra-Scorer parameter)

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \]

where \( U \) is the horizontal wind speed in the undisturbed motion,

\( w \) is the vertical velocity,

\( S = g \theta^{-1} \frac{\partial \theta}{\partial z} \) the stability,

\( \theta \) the potential temperature and \( g \) the acceleration of gravity.

Palm and Foldvik find a solution to equation (1) for a surface streamline vertical displacement \( \zeta_0 \), corresponding to a mountain profile \( \zeta_0 = a \cos kx \), at \( z = 0 \) with "a" a constant and \( k \) the wave number.

If \( \omega \) is a harmonic function of \( x \), equation (1) takes the form

\[ \frac{\partial^2 \omega}{\partial z^2} + (\zeta^2 - k^2) \omega = 0 \quad \ldots (2) \]

with boundary conditions

\[ \omega = U(0) \frac{\partial \zeta}{\partial x} = -a U(0) k \sin kx \text{ at } z = 0 \]

\[ \omega = 0 \text{ at } z = \infty \]

Scorer (1962) expresses equation (2) as approximately

\[ \frac{\partial^2 \zeta}{\partial z^2} + (\zeta^2 - k^2) \zeta = 0 \]

where \( \zeta \) is the vertical displacement of a particle from its original level and \( 2\pi/k \) the lee wave length.
For the cases in which this equation is roughly correct Scorer (1962) says that if there are lee waves it is essential that $k = l$ at some level while rotors will occur in the lee waves at these levels

$$\frac{\partial \zeta}{\partial z} > 1 \quad \text{for} \quad \zeta > 0$$

or

$$\frac{\partial \zeta}{\partial z} < -1 \quad \text{for} \quad \zeta < 0$$

Rotors can also occur at the nodal surfaces $\zeta = 0$.

Calculation of the $\zeta$ distribution is outside the scope of this paper, so the possibility of rotors will not be investigated.

3. DATA ANALYSIS

The $I^2$ distributions calculated from the 2300 GMT radiosonde data at Moree on 27 and 28 July, using the method of Spillane and Colquhoun (1966), are shown in Figure 1.

The winds were resolved in the direction $230^\circ$, as the axis of the mountain ridge between Brisbane Airport (27° 26'S, 153° 05'E), the radiosonde station on the lee side of the barrier, and Moree is normal to this direction. The $I^2$ distributions on both days have forms favourable for standing waves, i.e. large values in the lower levels decreasing steadily with increasing height. The profiles also satisfy Palm and Foldvik's (1960) condition for lee waves that $I$ at the ground should be more then 1.5 times the first minimum value in the $I$ profile.

The Brisbane Airport radiosonde traces (see Fig. 2) on the same days show evidence of standing waves, especially the 2300 GMT trace on 28 July which evidences markedly periodic perturbations. The balloon appears to have travelled through one wavelength between the consecutive points marked on the traces in Fig. 2. From the ascent rates of the balloon and the average wind speeds between the points the average wavelengths were found to be 12.4 n mi on 27 July and 3.5 n mi on 28 July.

More substantial evidence of the existence of standing waves and more accurate estimates of wavelengths can be obtained from radiosonde recorder traces (Reid (1952) and Andersen (1966)). Figure 3 gives the altitude time graph for the Brisbane Airport sonde flights at 2300 GMT on 27 and 28 July. Also shown is the vertical velocity of the air (ft/min), calculated as departures from the average ascent rate of the balloon over half minute intervals. A graph of smoothed vertical velocity is also drawn based on running means of three points. Both of the graphs of smoothed vertical velocity against time show marked periodicity. Average wavelengths are 9.9 n mi on 27 July and 5 n mi on 28 July.

Comparison of the Moree and Brisbane Airport traces, Figs. 1(b) and 2(b), reveals that the air at Moree between 750 mb and 860 mb had descended and warmed considerably on reaching Brisbane Airport. A descending phase of the lee wave above Brisbane Airport is the probable cause of this warming. Fig. 3(b) shows strong down motion between 1500 and 3500 ft over Brisbane Airport.

4. THEORETICAL ESTIMATES OF WAVELENGTHS AND MAXIMUM VERTICAL VELOCITIES

Foldvik (1962) gives a method of evaluating wavelengths and maximum vertical velocities from the $I^2$ distribution of the undisturbed airstream. Approximating the observed $I$ curve by the exponential function of $z$. 
\[ \ell = \ell (z) = \ell (0) e^{-cz} \]  \hspace{1cm} \cdots (3)

where \( \ell (0) \) and \( c \) are constants.

Foldvik gives the wave solution of equation (1) and states that the wave numbers, \( k_T \), of the resonant waves (if any) are determined by

\[ J_{k_T/c} \left( \frac{\ell (0)}{c} \right) = 0 \]

where \( J \) denotes the Bessel function of the first kind.

Foldvik presents a diagram showing the dependence of wavelength on \( \ell (0) \) and \( c \). Depending on the value of \( \ell (0)/c \) no waves, one wave system or two or more wave systems may exist. He also states that if \( \ell (0)/c \) is greater than 8.6 three or more wave systems may exist, but these are not shown in his diagram because such high values of \( \ell (0)/c \) seldom occur.

Figure 4 shows the \( \ell \) distribution with height at Moree at 2300 GMT on 27 and 28 July. The \( \ell \) distribution was calculated using Foldvik's procedure of smoothing the temperature and wind profiles and calculating \( \ell^2 \) over 100 mb layers. The term \( -1 \frac{d^2U}{dx^2} \) in the expression for \( \ell^2 \) is neglected. Following Foldvik, exponential approximations were fitted between the ground and the first minimum in \( \ell \). On 28 July approximation (1) (see Fig. 4) fulfills this condition.

From the exponential approximations, \( \ell (0) \) and \( c \) values of 1.74 km\(^{-1} \) and 0.329 km\(^{-1} \) on the 27th and of 1.21 km\(^{-1} \) and 0.264 km\(^{-1} \) on the 28th were obtained. With both of these sets of values only short wavelength waves, with values of 5.6 n mi on the 27th and 9.3 n mi on the 28th, are possible.

In the shortest wavelength system the height, \( h_1 \), of the maximum vertical velocity is given by

\[ h_1 = c^{-1} \ln \frac{\ell (0)}{\ell (0) - 2.2 c} \]  \hspace{1cm} \cdots (4)

and the maximum vertical velocity, \( w_1 \) \( \max \), by

\[ w_1 \max = (2.5 + 0.7) \frac{c}{cL_1} U \theta (0) \left( \frac{\theta (0)}{\rho (h_1)} \right)^{1/2} \]  \hspace{1cm} \cdots (5)

where \( L_1 \) is the wavelength, \( U (0) \) the horizontal wind velocity at the mountain height, \( H \) the height of the mountain and \( \rho \) the density of the air.

Using equation (4), heights of maximum vertical velocity of 5300 ft on the 27th and 8100 ft on the 28th are obtained.

The maximum vertical velocity is a function of the height, \( H \), of the barrier over which the air streams. Caswell (1966) states that the lee slope is generally recognized as the most important factor of a mountain profile. \( H \), according to Caswell, should not be taken as the height of the mountain peak above mean sea level but the difference between the general height of the mountain barrier in the vicinity of its lee slope and the height of the ground to its lee.

The elevation profile in the direction 230\(^\circ\) from Brisbane Airport (see Fig. 5) is a complex one. The mountain height is readily obtained but it is difficult to determine the lee slope height to be used. The height actually used (1900 ft) is that marked in Fig. 5.
Fig. 1(a) — Temperature and smoothed temperature soundings, wind profiles and $\zeta^2$ distributions at Moree at 2300 GMT on 27 July 1966.
Fig. 1(b) — Temperature and smoothed temperature soundings, wind profiles and $e^2$ distributions at Moree at 2300 GMT on 28 July 1966.
Fig. 2(a) Temperature sounding at Brisbane Airport
at 2300 GMT on 27 July 1966.
Fig. 2(b) Temperature sounding, vertical wind shears and Richardson number values at Brisbane Airport at 2300 GMT on 28 July 1966.
Fig. 3(a) — Graphs of altitude of radiosonde against time, vertical velocity of the air and smoothed vertical velocity of the air derived from radiosonde flights at Brisbane Airport at 2300 GMT on 27 July 1966.
Fig. 3(b) — Graphs of altitude of radiosonde against time, vertical velocity of the air and smoothed vertical velocity of the air derived from radiosonde flights at Brisbane Airport at 2300 GMT on 28 July 1966.
Fig. 4 — Foldvik's approximation to the $\ell$ curve at Moree at 2300 GMT on 27 and 28 July 1966.

Fig. 5 — Elevation profile in the direction 230° from Brisbane Airport. 'H' is the height of the barrier above the general height of the ground to the lee of the barrier.
Using \( U(o) \) and \( (Q(o)/Q(h1))^\frac{1}{2} \) values of 28 kt and 1.08 on the 27th and 30 kt and 1.14 on the 28th in equation (5), maximum vertical velocities of 1600 ft/min on the 27th and 1420 ft/min on the 28th are obtained.

To specify the maximum vertical velocity completely, its distance from the top of the barrier is required. Palm and Foldvik (1960) give a theoretical vertical velocity distribution in which the maxima of vertical velocity occur at horizontal distances, from an origin on the lee side and close to the top of the barrier, of integral multiples of half the wavelength of the lee wave. In their diagram these maxima have the same value. In practice, however, it seems likely that waves will attenuate and disperse, thus producing a maximum vertical velocity close to the barrier. Difficulty exists in determining the position of the origin for a non-symmetrical barrier and hence a reliable estimate of the horizontal position of the maximum vertical velocity cannot be made.

Caswell modified Foldvik's method. He eliminated the \( L \)-profile calculation by representing the atmosphere by two layers, 1000 to 700 mb and 700 to 500 mb, and obtained an approximation to the \( L \)-profile by using the exponential approximation (equation (3)) made to fit the observed \( L \)-values found from the layers centred on 850 mb and 500 mb. The two \( L \)-values, \( L_{850} \) and \( L_{500} \), are found graphically using the temperature differences over each layer and their mean wind speeds.

\( L_{850} \) values of .89 and .93 km\(^{-1} \) and \( L_{500} \) values of .47 and .37 km\(^{-1} \) were found for 27 and 28 July respectively. Using these \( L \)-values in Caswell's Figs. 3 and 4, it is seen that two waves can exist on the 27th and one on the 28th. From Caswell's diagrams the wavelength, height of maximum vertical velocity and \( C \) values can be obtained.

For the shorter wavelength system, \( C_1 \) is defined by

\[
C_1 = (2.5 + \frac{0.7}{cL_1}) c \frac{Q(o)}{Q(h1)}\frac{1}{2}
\]

Hence from equation (5)

\[
|w_1|_{max} = H U(o) C_1
\]

For the longer wavelength system

\[
C_2 = 3.2 c \frac{Q(o)}{Q(h2)}\frac{1}{2}
\]

and

\[
|w_2|_{max} = H U(o) C_2
\]

Table 1 compares the values of wavelengths, heights of maximum vertical velocities and maximum vertical velocities found by the various methods.

From the first exponential approximation (see Fig. 4) on 28 July, Foldvik's method gives a large wavelength value. The second approximation was derived by considering the distribution between the first maximum and the first minimum of \( L \). With \( L(o) \), \( c \), and \( (Q(o)/Q(h1))\frac{1}{2} \) values of 1.54 km\(^{-1} \), 313 km\(^{-1} \) and 1.11 respectively, the wavelength derived from this approximation is close to that given by Caswell's method.
Table 1. Comparison of experimentally and theoretically determined values of lee wave parameters, where L is the wavelength, $h_m$ the height of the maximum vertical velocity and $w_m$ the maximum vertical velocity.

<table>
<thead>
<tr>
<th>Method</th>
<th>27 July 1966</th>
<th>28 July 1966</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L (n mi)</td>
<td>$h_m$ (ft)</td>
</tr>
<tr>
<td>Radiosonde Trace</td>
<td>12.4</td>
<td>3.5</td>
</tr>
<tr>
<td>Vertical Velocity Distribution</td>
<td>9.9</td>
<td>15,000</td>
</tr>
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<td></td>
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<tr>
<td>Foldvik Method</td>
<td></td>
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<tr>
<td>Approx. (1)</td>
<td>5.6</td>
<td>5,320</td>
</tr>
<tr>
<td>Approx. (2)</td>
<td></td>
<td>6.1</td>
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<tr>
<td>Caswell Method</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First wave</td>
<td>6.1</td>
<td>7,100</td>
</tr>
<tr>
<td>Second wave</td>
<td>30</td>
<td>34,000</td>
</tr>
</tbody>
</table>

The maximum vertical velocities and heights of maximum vertical velocities determined from the vertical velocity distribution and by Foldvik's and Caswell's methods cannot be compared because the radiosonde may not have passed through a region of maximum vertical velocity. Also Brisbane Airport is some 58 miles from the summit of the mountain barrier, so it seems likely that the waves would have been attenuated at Brisbane Airport and the maximum vertical velocity sampled by the radiosonde probably would have been considerably less than the ones estimated using the theoretical methods.

A test of the compatibility of Foldvik's and Caswell's methods was made by finding $\lambda_{850}$ and $\lambda_{500}$ from the exponential approximation of Foldvik and using Caswell's graphs to determine wavelengths and vertical velocity parameters. Wavelengths and heights of the level of maximum vertical velocity found by this procedure are slightly smaller than those found by using Foldvik's method, because Caswell's method derives the $\lambda(o)$ value at a fixed level and not at station level, resulting in this case in an overestimate of $\lambda(o)$ and an underestimate of the wavelength. However, this does not affect the values of the maximum vertical velocities.

5. CONCLUSIONS

Marked evidence of the existence of lee waves is seen in the analysis of the radiosonde data at Brisbane Airport. It is not possible to assess the merits of the Foldvik and Caswell methods of estimating vertical velocity, as the radiosonde may not have sampled the air in the region of maximum vertical velocity. The Caswell method gave maximum vertical velocities about 400 ft/min less than those obtained by Foldvik method.

The theoretical methods give similar wavelength values, but the Caswell method is preferable because of its operational simplicity and the difficulty sometimes encountered in deciding upon the best exponential approximation to be used in the Foldvik method.
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REFERENCES


