APPLICATION OF THE LINEAR DISCRETE FILTER TO THE PROCESSING OF METEOROLOGICAL ROCKET WIND DATA

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ABSTRACT

A number of meteorological rockets with dropsonde and metallised parachute payloads were fired at Woomera, South Australia in 1968.

The parachutes were tracked by the Commonwealth Bureau of Meteorology's WF 44 radar to permit determination of temperature profiles (using the telemetered temperature data) and wind profiles. Smoothed position, velocity and acceleration data were required in order to allow for poor wind sensor response at the highest altitudes. Smoothing was achieved using the Linear Discrete Filter which is based on the assumption that the acceleration is constant.

Some properties of this filter are discussed and error estimates for the smoothed data are derived. It is shown that with smoothing extending over about three-quarters of a minute and with an input rate of one point per second, satisfactory profiles can be determined with an accuracy of a few meters per second.

INTRODUCTION

During the period August to November 1968, eighteen rocket dropsonde experiments were carried out jointly by the Bureau of Meteorology and the Weapons Research Establishment (WRE) at Woomera. The primary purpose of the soundings was to compare the accuracies of two different dropsonde systems.

A secondary aim was to determine the extent to which the Bureau of Meteorology WF 44 windfinding radar would be suitable for dropsonde purposes. In its normal balloon windfinding role the WF 44 is a modern radar of adequate tracking accuracy, and with provision for variable data output rates, with a maximum of one point every six seconds.

In dropsonde procedures the tracking requirements are considerably more stringent due to factors of the following types which arise in the higher sounding altitudes:

(a) The initial tracking range is always large, e.g. 80 km because of the altitudes reached and because of the distance from the radar to the launching area.

(b) The descent rates of the parachute are substantial, and change fairly rapidly with altitude (e.g. typical values are 280 m s\(^{-1}\) at 70 km, 125 m s\(^{-1}\) at 60 km and 26 m s\(^{-1}\) at 40 km.

(c) Since the dropsonde does not include a barometric sensor, the altitude must be determined from the radar positional data.
(d) The temperature sensor is subject to corrections for dynamic heating. These corrections are functions of the square of the total velocity relative to the air. Typical corrections are 18°C at 65 km, 8°C at 60 km and 2°C at 50 km.

(e) The response of the parachute to wind is a function of the ambient air density, and is relatively poor at high altitudes. The observed displacement of the parachute therefore does not determine the true wind speed and direction. This problem appears in the most acute form immediately after the dropsonde has been ejected from the rocket. At that stage the rocket (and parachute) forward velocity is substantial, e.g., of the order of 100 m s⁻¹. Assuming that a rocket is fired towards the west, the parachute will continue moving in that direction after ejection. The observed initial displacement of the parachute will therefore suggest spurious easterly winds, even though the true winds at the same altitude may be blowing from the west. To remove or minimise this sort of error it is necessary to take the acceleration terms into account. The measurement of acceleration is obviously a major problem, since it involves double differentiation or taking second differences in the positional data which are not very smooth.

Because of the requirement for highly accurate trajectory data, it is customary to use a radar such as the FPS16, which has better angular accuracy than the WF 44, and which provides for data output at rates as high as 40 points per second. In this case it is possible to use comparatively simple data filters over relatively short segments of the trajectory. The same filters are not satisfactory for the coarser WF 44 data. Several alternative methods were examined, and the results compared with data obtained from FPS16 radars which also tracked the dropsondes. The WF 44 was modified to give data output at four points per second.

The purpose here is to describe the type of filter finally adopted, and its characteristics.

**CHOICE OF FILTER**

When the question of the appropriate filter was being considered, it was anticipated that the raw cartesian data would contain substantial small scale detail; some were real (and due to such causes as true small-scale wind variability, aerodynamic behaviour e.g. "gliding" of the parachute and dropsonde assembly), some were associated with the basic characteristics of the radar, and some arose from the truncation of the radar angular data to the nearest one-tenth of a degree. Since the extent to which these effects would be present was not well known, it was necessary after trials data were available to decide on a particular filtering interval mainly by comparison of the wind data with that produced by the more accurate FPS16 radar.

To obtain useful results it was clearly necessary to filter the cartesian data. If \( x_i \) is the resolved part of the slant range on the X axis (there will be similar points \( y_i \) and \( z_i \) on the Y and Z axes respectively) the problem is to choose a suitable filter that can be applied to a set of \( N \) points \( x_1', x_2', \ldots, x_N \) so that smoothed values of \( x \) can be determined together with the first and second derivative (velocity and acceleration) either directly or indirectly.

The following three relations may be fitted using least squares:

\[(i) \quad x_i' = a\]
\[(ii) \quad x_i' = a + bt_i\]
\[(iii) \quad x_i' = a + bt_i + ct_i^2\]

where: \( t \) refers to time
\( a, b, c \), are constants.
It is clear that in general the coordinates $x_i$, $y_i$ and $z_i$ will be varying over a given time interval. In particular in the case of the $Z$ coordinate even the first derivative ($\dot{z}$, the vertical velocity) will not be constant because the assembly rapidly descends into denser air giving a decreasing velocity. Also, since the derivatives of the positional coordinates $x_i$, $y_i$ are closely related to the horizontal wind which varies with height, both $\dot{x}_i$ and $\dot{y}_i$ will vary with height. It was therefore decided to construct a filter based on the assumption that the variation of $x_i$, $y_i$ and $z_i$ could be expressed satisfactorily by a second order polynomial (quadratic) with respect to time.

A quadratic filter of this type makes the not very stringent assumption that acceleration is constant over the smoothing interval. Because provision is made for acceleration, less real detail should be suppressed than by using either a running mean or linear filter. This assumption was subsequently confirmed in a more detailed study by Johnson (1968). Filters whose weights are based on the probability or exponential curves have an advantage in that their responses are always positive but they also suppress detail. Cubic filters were also considered but were found to give only a small increase in accuracy with an undesirable tendency to become computationally unstable.

Following previous practice (Knight, 1962, Hynes, 1959) the quadratic based filter will be referred to as the Linear Discrete Filter.

**Mathematical Aspects of Filter**

Let $x_i$ be the value of a positional coordinate at time $t_i$.

Let $A$, $B$, $C$ be constants.

Assume that $x_i$ may be suitably represented by a quadratic in $t$, i.e.

$$x_i = B + At_i + Ct_i^2 + \epsilon_i$$

where $\epsilon_i$ is an error term.

Calculating the constants $B$, $A$, $C$ by least squares, a smoothed positional value $x^*$ may be defined where:

$$x^* = B + At_m + Ct_m^2$$

where $t_m$ is the mid-value of the $t$'s.

If the observations are taken at uniform intervals and the $t$'s are centred around zero, it may be shown (Appendix I) that the constants $B$, $A$, $C$ are linear functions of the $x_i$'s and the interval "h" between the observations, i.e.

$$x^* = B = \Sigma_{j=0}^{(N-1)/2} B_j x_j$$

where $N$ is the number (odd) of observations and $B_j$ is a set of predetermined weights.
Similarly for velocity and acceleration it is shown that smoothed velocity \( \dot{x}^* \) is given by the expression

\[
\dot{x}^* = A = \frac{1}{h} \sum_{j=-(N-1)/2}^{(N-1)/2} \alpha_j x_j
\]

and smoothed acceleration \( \ddot{x}^* \) by the equation

\[
\ddot{x}^* = C = \frac{1}{h^2} \sum_{j=-(N-1)/2}^{(N-1)/2} \gamma_j x_j
\]

where \( \alpha_j \) and \( \gamma_j \) are also sets of predetermined weights.

It is because the position, velocity and acceleration may be derived as linear functions of the \( x_j \) that the filter is so named.

**ERROR ANALYSES**

Using the Linear Discrete Filter, it may be shown that slightly more accurate smoothed positional data result from using \( N \) even, and slightly more accurate smoothed velocity data result from using \( N \) odd. The differences are insignificant for large \( N \) (say, greater than 10) and as it is simpler to use \( N \) odd, this was done in the following description of the processing scheme and error analysis.

In balloon windfinding the radar target is usually a "corner reflector" which has substantial "gain", and provides a signal which is both reasonable and largely independent of target attitude. By comparison the signal/noise ratio with a parachute target is variable, and at times poor. Any large spurious signals from this source would diminish the effectiveness of the filter, and could introduce unacceptable errors. Some form of quality control was therefore necessary.

Initially the Linear Discrete Filter was applied to the first \( "N" \) points. After derivation of the smoothed position, velocity and acceleration terms at the midpoint, the same curve was used to predict the probable coordinates of the \( "N+1" \) point. These were compared with the actual values of the \( "N+1" \) points, and if the difference exceeded a certain amount, the actual values were replaced by the predicted coordinates. To distinguish between random fluctuations and true discontinuities, only a limited number of consecutive substitutions was permitted.

The calculations were carried out on a digital computer and details of the program are available from the author.

The above filtering and quality control procedure was applied on a continuous basis in the same manner as a running mean. If \( \sigma_x \) is the standard error in \( x \), and \( h \) is the time interval between observations and assuming initially that the intercorrelation between the input data points is zero, then the standard errors of the smoothed position \( \sigma_x^* \), velocity \( \sigma_x^* \) and acceleration \( \sigma_x^{**} \) are given by (see Appendix I)

\[
\sigma_x^* = \sqrt{\frac{3(3N^2-7)}{4N(N^2-4)}} \cdot \sigma_x
\]

\[
\sigma_x^* = \sqrt{\frac{12}{N(N^2-1)}} \cdot \sigma_x
\]

\[
\sigma_x^{**} = \sqrt{\frac{180}{N(N^2-1)(N^2-4)}} \cdot \frac{2\sigma_x}{h^2}
\]
Now the radar errors arise from:

(a) random and systematic errors in azimuth, elevation and slant range
(b) truncation errors in azimuth, elevation and to a much lesser degree, slant range.

To assess the likely magnitude of these errors (excluding systematic errors) the differences between the expected and actual points were noted and the rms values of the differences formed. Although these rms values varied widely the following figures were taken as being reasonably representative:

\[
\begin{align*}
\text{rms error in slant range} & \quad 8 \text{ m} \\
\text{" " azimuth} & \quad 0.1^\circ \\
\text{" " elevation} & \quad 0.15^\circ 
\end{align*}
\]

In its present form the radar data are truncated to the nearest 10 m in slant range and \(0.1^\circ\) in elevation and azimuth.

Taking a representative slant range of 75 km and evaluating the equivalent rms elevation and azimuth errors, it is seen that the error in range is only about 5 to 10 percent of that due to elevation and azimuth errors. The rms error shape resembles a volume made up of two elliptical saucer shapes with the slant range vector parallel to the smallest axis of the volume. In the coordinate system used, the projections along the three axes (east-west, north-south, and vertical) were similar in an order of magnitude sense since the initial elevations were about 40° and the initial bear about 310°. Consequently it was decided for simplicity to distribute the rms errors in slant range, azimuth and elevation equally along these three cartesian axes, i.e. denoting \(\sigma_x\) as the rms error in the \(x\) direction etc., then

\[
\sigma_x = \sigma_y = \sigma_z = 145 \text{ m}
\]

Using this value in the above expressions for the standard errors of position, velocity and acceleration gives for various smoothing intervals and various input data rates, the estimates of error of position, velocity and acceleration of the assembly, as given in Tables 1 to 3.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Computed standard errors (m) in each positional component of the assembly</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Based on (\sigma_x = 145 \text{ m}))</td>
<td></td>
</tr>
<tr>
<td>(The number of points involved in the smoothing is given in brackets to the right of the standard error)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input data interval (s)</th>
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<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smoothing period (s)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>43.9 (25)</td>
<td>84.3 (7)</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>146.0 (3)</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>28.0 (61)</td>
<td>54.7 (16)</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td>108.3 (4)</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>19.9 (121)</td>
<td>39.4 (31)</td>
<td>89.0 (6)</td>
</tr>
<tr>
<td>42</td>
<td></td>
<td>77.2 (8)</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>16.3 (181)</td>
<td>32.3 (46)</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td></td>
<td>73.8 (9)</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>14.1 (241)</td>
<td>28.0 (61)</td>
<td>66.5 (11)</td>
</tr>
</tbody>
</table>
Table 2 Computed standard errors in velocity (m s^{-1}) of the assembly

(Based on $\sigma_v = 145$ m)
(The number of points involved in the smoothing is given in brackets to the right of the standard error)

<table>
<thead>
<tr>
<th>Smoothing period (s)</th>
<th>Input data interval (s)</th>
<th>0.25</th>
<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td>16.20 (25)</td>
<td>27.60 (7)</td>
<td></td>
</tr>
<tr>
<td>12</td>
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<td>17.21 (3)</td>
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<td>15</td>
<td></td>
<td>4.25 (61)</td>
<td>8.15 (16)</td>
<td></td>
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<tr>
<td>18</td>
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<td></td>
<td>16.32 (4)</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>1.52 (121)</td>
<td>2.93 (31)</td>
<td>7.05 (6)</td>
</tr>
<tr>
<td>42</td>
<td></td>
<td></td>
<td>4.20 (8)</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
<td>0.83 (181)</td>
<td>1.63 (46)</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td></td>
<td></td>
<td>3.14 (9)</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td>0.54 (241)</td>
<td>1.06 (61)</td>
<td>2.32 (11)</td>
</tr>
</tbody>
</table>

Table 3 Computed standard errors in acceleration (m s^{-2}) of the assembly

(Based on $\sigma_v = 145$ m)
(The number of points involved in the smoothing is given in brackets to the right of the standard error)

<table>
<thead>
<tr>
<th>Smoothing period (s)</th>
<th>Input data interval (s)</th>
<th>0.25</th>
<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td>20.14 (25)</td>
<td>31.85 (7)</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td>9.93 (3)</td>
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<tr>
<td>15</td>
<td></td>
<td>2.16 (61)</td>
<td>3.86 (16)</td>
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<td>18</td>
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<td>4.06 (4)</td>
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<td>0.63 (8)</td>
<td></td>
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<tr>
<td>45</td>
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<td></td>
<td>0.46 (9)</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td>0.070 (241)</td>
<td>0.13 (61)</td>
<td>0.28 (11)</td>
</tr>
</tbody>
</table>
Fig. 1  Velocity accuracy as a function of smoothing period for various input data intervals ($\Delta t$) and two serial correlation ($R_t$) cases.
In practice it was found that the data were serially correlated. To assess
the extent of this serial correlation, radar data at \( \frac{1}{6} \) s intervals (in polar coordinate
form) for one firing of a Skua 108 on 3 December, 1968 were transformed into cartesian
axes (east-west, north-south and vertical) "prewhitened" by using the Linear Discrete
Filter and serial correlation coefficients evaluated for each component from a set
of 7000 points for lags from 1 point to 100 points. The calculations were carried
out for smoothing periods ranging from 6 to 60 s.

As expected the correlation coefficients decreased as the lag increased and
the relationship could be analytically expressed with sufficient accuracy by the
equation:

\[ R = \exp(-kt) \]

where:  \( t \) is the time lag
\( R \) is the correlation coefficient for a lag \( t \)
\( k \) is a constant which was determined from graphical plots on semi-log paper.
This constant varied from about 0.9 to 1.5.

It is shown in Appendix I that the presence of serial correlation in the
data has the effect of decreasing the accuracy of the derived winds. To estimate this
loss of accuracy, a conservative value of \( k = 1.1 \) was chosen and standard errors of the
computed winds were derived using the expression:

\[ R = \exp(-1.1 \times t) \]

for the serial correlation.

These estimates, and also estimates based on no serial correlation, are
shown graphically in Fig 1 to 3. Regarding \( 1/k \) as a "time constant" of the data
(the value of \( R \) drops to 1/e when \( t = 1/k \)), it is seen from the graph that for data
input rates at 1 point each 6 s the serial correlation has negligible effect on the
accuracy. However for an input rate of 1 point per s, the serial correlation reduces
the accuracy of the position, velocity and acceleration estimates by between 20 and 30
percent. It is clear also that there is no advantage gained by using data at \( \frac{1}{6} \) s
intervals rather than 1 s intervals.

The dropsonde parachute assembly becomes an increasingly poor wind sensor
with height and above about 40 km it is necessary to make what may be regarded as slip
corrections to the measured horizontal components of motion of the assembly to obtain
the true wind components. Appendix II contains an analysis of the error of this
correction. It is shown that the standard error of a computed wind component \( (\sigma_j, \sigma_y) \)
is the vector sum of the standard error of the observed horizontal velocity \( (\sigma_x, \sigma_y) \)
and the standard error of the slip correction \( (\sigma_z) \).

Estimates of \( q_m \) have been made for the three levels 40, 50 and 60 km for
a number of dropsonde-parachute descent trajectories and these are plotted in Fig 4.
These estimates of \( q_m \) were derived from the observed velocities and accelerations and
the estimates of the standard errors in velocity and acceleration given earlier. There
were two systems used; the English Skua and the Australian Kookaburra meteorological
rockets.

In practice it was found that smoothing extending over a period of 44 s
(together with a data input rate of 1 point per s) gave satisfactory wind data. In this
case it is seen from Fig 1 that the standard error of the horizontal velocity of the
assembly, allowing for serial correlation was about 2.3 m s\(^{-1}\) in each component. This
was combined with the standard error of the wind correction term to give error
estimates of the final wind components. These results are given in Table 4.
Fig. 2  Height accuracy (metres) as a function of smoothing period for various input data intervals ($\Delta t$) and two serial correlation ($R_1$) cases.
Fig. 3  Acceleration accuracy as a function of smoothing period for various input data intervals ($\Delta t$) and two serial correlation ($R_t$) cases.
STANDARD ERROR OF WIND CORRECTION FACTOR $\sigma_u$ (metres per second$^{-1}$)

Note: The groups of factor $\sigma_u$ are at the same vertical height though displaced vertically for clarity.

Fig. 4 Estimates of the standard error of the slip correction factor $\alpha_u$ (m s$^{-1}$) of the zonal and meridional components based on data from Woomera meteorological firings.
Table 4 Standard errors of the computed wind components (m s\(^{-1}\))

<table>
<thead>
<tr>
<th>Height (km)</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Errors of:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wind correction term</td>
<td>1.0</td>
<td>1.7</td>
<td>2.5</td>
</tr>
<tr>
<td>Horizontal velocity of the assembly</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
</tr>
<tr>
<td>Each computed wind component</td>
<td>2.5</td>
<td>2.9</td>
<td>3.4</td>
</tr>
</tbody>
</table>

**RESPONSE OF LINEAR DISCRETE FILTER**

The response of mathematical filters has been discussed by Holloway (1958) who has given expressions for evaluating the response of certain filters.

Defining the response \( R \) to be the ratio of the mean of the filtered series to that of the original series, it is shown that for an even weighting function,

\[
R(f) = \sum_{k=-n}^{m} w_k \cos 2\pi fk
\]

where: \( n + m + 1 = N \), the total number of weights
\( w_k \) is the \( k \)th weight
\( f \) is the frequency in terms of cycles per data interval.

Examples of the response of filters are given for two types of filter in Fig 5.

Firstly, there is the case of a filter with all the weights equal, i.e. a 'running mean' filter. Secondly, the response of the Linear Discrete Filter is given. The number of weights was taken to be the same in both cases to facilitate a comparison; the actual number was forty five. This was the number corresponding to the smoothing interval (44 s) which appeared from an examination of the smoothed wind profiles and the residual "noise" to give the most satisfactory results. For this smoothing interval, the winds were also reasonably consistent with winds derived from an FPS16 radar.

The main feature of the response curve of both filters is the presence of a relatively large periodic component. In the case of the Linear Discrete Filter this sometimes results in a phase reversal of components of frequency 1 cycle/25 s and higher. The amplitude of frequencies so affected will be reduced to 25 percent or less of their original value. It follows that the reality of any features less than about 25 s is questionable.

It is seen that the response of the Linear Discrete Filter first becomes zero at approximately 1.73 cycle/44 s which is about 1 cycle/25 s. The equal-weight filter has its first response zero at 1 cycle per smoothing interval. Hence one may look upon the Linear Discrete Filter of 44 s as being roughly equivalent (in terms of length of smoothing) to a 25 s running mean.

To illustrate the effect of the Linear Discrete Filter a sinusoidal input was subjected to the filter such that the filter smoothing interval covered firstly 1 cycle, secondly 1½ cycle and thirdly 2 cycle of the input wave. The input curve and the three filtered output curves are shown in Fig 6 (a, b, c). It will be seen from the response curve that the responses should be 0.76, 0.24 and - 0.19. These responses and the phase reversal in the third case are in accordance with previous conclusions.
Fig. 5  Filter responses.
Fig. 6 The effect of the linear discrete filter on a sinusoidal input wave.
CONCLUSIONS

The investigation described above indicates that the WF 44 radar can be used for dropsonde purposes, providing the rate of data output is increased, and a suitable method of data reduction is employed.

The Linear Discrete Filter was adopted, as it provides better accuracy and detail than straight line fitting, probably because the assumption of constant acceleration over the smoothing interval represents the physics of the motion tolerably well. A quadratic fit was also found to be less subject to instability than curves of higher order.

Using the Linear Discrete Filter, the error at 60 km in each wind component is about 4 m s$^{-1}$, and this might be regarded as the upper altitude limit of the WF 44 unit for data of acceptable accuracy. Higher altitudes could be achieved by improving the target characteristics of the parachute.

REFERENCES


APPENDIX 1

Derivation of the Linear Discrete Filter

The object of the Linear Discrete Filter is to take a set of N observations of a variable \( x \), equally spaced in time (t) by an increment h, and to compute a smoothed value \( x' \) of the variable at the midpoint together with the smoothed velocity \( (\dot{x}') \) and acceleration \( (\ddot{x}') \) at the midpoint.

In fitting a meaningful curve to the data, the simplest realistic assumption we can make is that the acceleration is non-zero and constant. This assumption defines a parabola whose equation will be of the form

\[
x = B + At + Ct^2
\]

where: A, B, and C are constants.

Defining \( m = \frac{N - 1}{2} \), we can, without loss of generality allow "t" to range from \(-mh\) to \(+mh\) and the equation above may be set out as

\[
x = B + Ahj + Ch^2j^2
\]

where: \( j \) takes the \( N \) values \(-m\) to \(+m\).

For convenience this may be rewritten as

\[
x = b + aj + cj^2
\]

where:
\[
b = B \\
a = Ah \\
c = Ch^2
\]

Solving by least squares we get the three equations:

\[
\sum x = Nb + a\sum j + c\sum j^2
\]

\[
\sum xj = b\sum j + a\sum j^2 + c\sum j^3
\]

\[
\sum xj^2 = b\sum j^2 + a\sum j^3 + c\sum j^4
\]

Now
\[
\sum j = 0
\]

\[
\sum j^2 = N(N^2 - 1)/12
\]

\[
\sum j^3 = 0
\]

\[
\sum j^4 = \frac{N}{240} (3N^2 - 7)(N^2 - 1)
\]

where the summation ranges from \(-m\) to \(+m\) in each case (and \( 2m + 1 = N \)).
Making the above substitutions and going through the algebra it is found that

\[ b = \sum_{j=-m}^{m} \beta_j x_j \]
\[ a = \sum_{j=-m}^{m} \alpha_j x_j \]
\[ c = \sum_{j=-m}^{m} \gamma_j x_j \]

where

\[ \beta_j = \frac{-601^2 + 3(3N^2 - 4)}{4N(N^2 - 4)} \]
\[ \alpha_j = \frac{121}{N(N^2 - 1)} \]
\[ \gamma_j = \frac{1801^2}{N(N^2 - 1)(N^2 - 4)} - \frac{15}{N(N^2 - 4)} \]

The smoothed position \( x^* \) is defined to be the mid-point value (\( j=0 \)) calculated from the quadratic; i.e. \( x^* = \beta = b \).

The smoothed mean velocity \( \dot{x}^* \) is defined to be the value of the first derivative of the quadratic calculated at the mid-point, i.e. \( \dot{x}^* = A = a/h \).

The smoothed mean acceleration \( \ddot{x}^* \) is defined to be the value of the second derivative of the quadratic calculated at the mid-point, i.e. \( \ddot{x}^* = 2C = 2c/h^2 \).

The values of the weights \( \beta, \alpha, \gamma \) are given for the case \( N=45 \) in Table A.

**Error analysis of the smoothed data**

(i) **Position**

As previously indicated, the mean position is given by

\[ x^* = b = \beta_{-m} x_{-m} \ldots \ldots + \beta_0 x_0 \ldots \ldots + \beta_m x_m \]

If \( x^*_i \) and \( x'_i \) denote deviations of a particular sample from the population mean, we have:

\[ x^*_i = + \beta_{-m} x'_{-m} + \ldots \ldots + \beta_0 x'_0 + \ldots \ldots + \beta_m x'_m \]
Table A  Linear Discrete Filter weights for N=45
for smoothed position ($\beta_1$), velocity ($\alpha_1$) and acceleration ($\gamma_1$)

<table>
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<tr>
<th>$\beta_1$</th>
<th>$\alpha_1$</th>
<th>$\gamma_1$</th>
</tr>
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<tr>
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<td>-0.00289855</td>
<td>0.00061671</td>
</tr>
<tr>
<td>-0.0269504</td>
<td>-0.00276680</td>
<td>0.00053262</td>
</tr>
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<td>0.00045243</td>
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<td>0.00037616</td>
</tr>
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<td>-0.00237154</td>
<td>0.00030379</td>
</tr>
<tr>
<td>0.00237506</td>
<td>-0.00223979</td>
<td>0.00023534</td>
</tr>
<tr>
<td>0.00781791</td>
<td>-0.00210804</td>
<td>0.00017080</td>
</tr>
<tr>
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<td>-0.00197628</td>
<td>0.00011017</td>
</tr>
<tr>
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<td>-0.00184453</td>
<td>0.00005346</td>
</tr>
<tr>
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<td>-0.00171278</td>
<td>0.00000065</td>
</tr>
<tr>
<td>0.02629062</td>
<td>-0.00158103</td>
<td>-0.00004824</td>
</tr>
<tr>
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<td>-0.00144928</td>
<td>-0.00009322</td>
</tr>
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<td>-0.00013429</td>
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<td>-0.00105402</td>
<td>-0.00020470</td>
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<td>-0.00092227</td>
<td>-0.00023404</td>
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<td>0.0289855</td>
<td>0.00061671</td>
</tr>
</tbody>
</table>
Squaring each side and summing over a set of samples we have:

$$
\Sigma x_i^* = \Sigma (\beta_{-m} x_{-m}^2 + \ldots + \beta_0 x_0^2 + \ldots + \beta_m x_m^2)
$$

$$
+ \Sigma (2\beta_{-m} \beta_{-(m-1)} x_{-m}^* x_{-(m-1)} + 2\beta_{-(m-1)} \beta_{-(m-2)} x_{-(m-1)} x_{-(m-2)} + \ldots + 2\beta_{m-1} \beta_m x_{m-1}^* x_m^*)
$$

$$
+ \Sigma (2\beta_{-(m-1)} \beta_{-(m-2)} x_{-(m-2)} x_{-(m-3)} + 2\beta_{-(m-1)} \beta_{-(m-3)} x_{-(m-1)} x_{-(m-3)} + \ldots + 2\beta_{m-2} \beta_m x_{m-2}^* x_m^*)
$$

$$
+ \ldots + \Sigma 2\beta_{-(m-2)} \beta_{-m} x_{-(m-2)} x_{-m}^*
$$

Now assuming that the values of $x^*$ form a stationary series, we have:

$$
\Sigma x^*_2 = \Sigma x_{-(m-1)}^2 = \Sigma x_{-(m-2)}^2 = \ldots = N_T \sigma_x^2
$$

where: $\sigma_x$ is the standard deviation of $x$ and $N_T$ is the total number of samples, and also:

$$
\Sigma x_{-(m-1)} x_{-(m-2)} = \Sigma x_{-(m-1)} x_{-(m-3)} = \ldots = N_T r_1 \sigma_x^2
$$

$$
\Sigma x_{-(m-2)} x_{-(m-3)} = \Sigma x_{-(m-1)} x_{-(m-3)} = \ldots = N_T r_2 \sigma_x^2 etc.
$$

where: $r_1$ is the auto-correlation coefficient between

$x_k^*$ and $x_k^* + 1$

Hence:

$$
\frac{\Sigma x^*_2}{N} = \sigma_x^2 (\beta_{-m}^2 + \beta_{-(m-1)}^2 + \ldots + \beta_m^2)
$$

$$
+ \sigma_x^2 (2r_1 (\beta_{-m} \beta_{-(m-1)} + \beta_{-(m-1)} \beta_{-(m-2)} + \ldots + \beta_{m-1} \beta_m))
$$

$$
+ \sigma_x^2 (2r_2 (\beta_{-(m-1)} \beta_{-(m-2)} + \beta_{-(m-1)} \beta_{-(m-3)} + \ldots + \beta_{m-2} \beta_m))
$$

$$
+ \ldots + \sigma_x^2 2r_m \beta_{-m} \beta_m
$$
i.e. \( \sigma_x^2 = \sigma_x^2 (\Sigma \beta_i^2) \left[ 1 + \left( \frac{2 \beta_1}{\Sigma \beta_i} \right) \left( \frac{m-1}{\Sigma j=-m} \beta_j \beta_{j+1} \right) \right. \\
+ \left. \frac{2 \beta_2}{\Sigma \beta_i} \left( \frac{m-2}{\Sigma j=-m} \beta_j \beta_{j+2} \right) \right. \\
+ \left. \frac{2 \beta_3}{\Sigma \beta_i} \left( \frac{m-3}{\Sigma j=-m} \beta_j \beta_{j+3} \right) \right. \\
\ldots+ \left. \frac{2 \beta_{2m}}{\Sigma \beta_i} \left( \frac{m-2m}{\Sigma j=-m} \beta_j \beta_{j+2m} \right) \right] \]

It is seen that the standard error of the smoothed position is a function of the standard error of the raw data \( \sigma_x \), the weights \( \beta_i \) and the intercorrelation factor \( R_B \) between the raw data points,

\[
\sigma_x^2 = \sigma_x^2 \left( \Sigma \beta_i^2 \right) \left[ 1 + R_B \right]
\]

where:

\[
R_B = \frac{2m}{\Sigma} \left( \frac{\beta_k}{m} \right) \left( \frac{m-k}{\Sigma j=-m} \beta_j \beta_{j+k} \right)
\]

Thus

\[
\beta_j = \frac{-60j^2 + 3(3N^2 - 7)}{4N(N^2 - 4)}
\]

\[
\Sigma \beta_i^2 = \frac{3600j^4 - 360j^2 (3N^2 - 7) + 9(3N^2 - 7)^2}{16N^2 (N^2 - 4)^2}
\]

Expressing \( \Sigma \beta_i^2 \) and \( \Sigma j^4 \) as functions of \( N \) and simplifying we obtain

\[
\Sigma \beta_i^2 = \frac{3(3N^2 - 7)}{4N(N^2 - 4)}
\]

Hence

\[
\sigma_x^* = \sigma_x \left[ 1 + \frac{\Sigma \beta_i^2}{4N(N^2 - 4)} \right]^{1/2}
\]

It is seen that (as expected) the lowest possible value of \( \sigma_x^* \) occurs when there is no serial correlation between the raw data points, that is, when all the data points are independent.
(ii) Velocity

Similarly, the standard error of the smoothed velocity is given by \( \sigma_x^* \) where

\[
\sigma_x^* = \frac{1}{h^2} \sigma_x \left[ \sum_{i=-m}^{m} a_i^2 \left[ 1 + R_a \right] \right]^{\frac{1}{2}}
\]

where

\[
R_a = \sum_{k=1}^{2m} \left[ \frac{2r_k}{m} \sum_{\substack{j=m \atop j=-m}}^{m-k} \sum_{i=-m}^{m} \alpha_i \alpha_{j+k} \right]
\]

\[
\Sigma_{\alpha_i}^2 = \frac{144 \Sigma_{\gamma_i}^2}{N^2 (N^2 - 1)^2} = \frac{12}{N(N^2 - 1)}
\]

hence

\[
\sigma_x^* = \frac{1}{h} \sqrt{\frac{12}{N(N^2 - 1)}} \cdot \sigma_x \left[ 1 + R_a \right]^{\frac{1}{2}}
\]

(iii) Acceleration

Again with the acceleration, the standard error of the smoothed acceleration \( \sigma_x^* \) is given by

\[
\sigma_x^* = \frac{4 \sigma_a^2}{h^4} = \frac{4 \sigma_x^2}{h^4} \left( \sum_{i=-m}^{m} \gamma_i^2 \right) \left[ 1 + R_Y \right]
\]

where

\[
R_Y = \sum_{k=1}^{2m} \left[ \frac{2r_k}{m} \sum_{\substack{j=m \atop j=-m}}^{m-k} \sum_{i=-m}^{m} \gamma_i \gamma_{j+k} \right]
\]

\[
\Sigma_{\gamma_i}^2 = \sum \left( \frac{180}{N(N^2 - 1)(N^2 - 4)} - \frac{15}{N(N^2 - 4)} \right)^2
\]

Expressing \( \Sigma_{\gamma_i}^2 \) and \( \Sigma_{\gamma_i}^4 \) as functions of \( N \) and simplifying we get

\[
\Sigma_{\gamma_i}^2 = \frac{180}{N(N^2 - 1)(N^2 - 4)}
\]

hence

\[
\sigma_x^* = \frac{2 \sigma_a}{h^2} \cdot \frac{180}{N(N^2 - 1)(N^2 - 4)} \cdot \sigma_x \left[ 1 + R_Y \right]^{\frac{1}{2}}
\]
(iv) Summary

Given a sample of \( N \) where \( N \) is an odd number of points \( x_i \) (where \( i = -m, \ldots, 0, \ldots, m; 2m + 1 = N \)) at uniform intervals of time (\( h \)), the Linear Discrete Filter weights \( \beta_j \), \( \alpha_j \), and \( \gamma_j \) and the expressions for smoothed position (\( x^* \)), velocity (\( \dot{x}^* \)) and acceleration (\( \ddot{x}^* \)) are as follows:

Position (\( x^* \)) = \( \sum_j \beta_j x_j \) where \( \beta_j = \frac{-60i^2 + 3(3N^2 - 7)}{4N(N^2 - 4)} \)

Velocity (\( \dot{x}^* \)) = \( h \sum_j \alpha_j x_j \) where \( \alpha_j = \frac{12i}{N(N^2 - 1)} \)

Acceleration (\( \ddot{x}^* \)) = \( \frac{2}{h^2} \sum_j \gamma_j x_j \) where \( \gamma_j = \frac{180i^2}{N(N^2 - 1)(N^2 - 4)} - \frac{15}{N(N^2 - 4)} \)

The standard errors of those smoothed quantities are related to the standard error of the raw data (\( \sigma_x \)) as follows:

\[
\sigma_{x^*} = \sqrt{\frac{3}{4N(N^2 - 4)}} \cdot \sigma_x \cdot \left[ 1 + R_{\beta} \right]^{1/2}
\]

\[
\sigma_{\dot{x}^*} = \frac{12}{N(N^2 - 1)} \cdot \sigma_x \cdot \left[ 1 + R_\alpha \right]^{1/2}
\]

\[
\sigma_{\ddot{x}^*} = \frac{2}{2h^2} \sqrt{\frac{180}{N(N^2 - 1)(N^2 - 4)}} \cdot \sigma_x \cdot \left[ 1 + R_\gamma \right]^{1/2}
\]

where

\[
R_{\beta} = 2m \sum_{k=1}^{m} \left[ \frac{2r_{k}}{m} \right] \sum_{i=-m}^{m} \beta_i \beta_{i+k}
\]

\[
R_{\alpha} = 2m \sum_{k=1}^{m} \left[ \frac{2r_{k}}{m} \right] \sum_{i=-m}^{m} \alpha_i \alpha_{i+k}
\]

\[
R_{\gamma} = 2m \sum_{k=1}^{m} \left[ \frac{2r_{k}}{m} \right] \sum_{i=-m}^{m} \gamma_i \gamma_{i+k}
\]

(Note that \( R_{\beta}, R_{\alpha}, \) and \( R_{\gamma} \) are functions only of the weights \( \beta, \alpha \) and \( \gamma \), and the serial correlation between the input data points).
APPENDIX 2

Accuracy of the derived winds

The two wind components are given (Kays and Olsen, 1967) by

\[ u = \dot{x} - \frac{\dot{z} \ddot{x}}{\ddot{z} + g} \]

\[ v = \dot{y} - \frac{\dot{z} \ddot{y}}{\ddot{z} + g} \]

where: \( \dot{x}, \dot{y}, \dot{z} \) are the three velocity components,

\( \dddot{x}, \dddot{y}, \dddot{z} \) are the three acceleration components,

\( g \) is gravity.

Differentiating the first equation, we have

\[ \Delta u = \Delta \dot{x} - \frac{\dot{z} \Delta \dddot{x}}{\ddot{z} + g} - \frac{\dddot{x} \Delta \dot{z}}{\dddot{z} + g} + \frac{\dot{z} \dddot{x} \Delta \dot{z}}{(\ddot{z} + g)^2} \]

Now squaring both sides, summing over the range of values and assuming that the cross correlations between the terms are negligible we get

\[ \sigma_u^2 = \sigma_x^2 + \sigma_c^2 \]

where

\[ \sigma_c^2 = \left( \frac{\dot{z}}{\ddot{z} + g} \right)^2 \sigma_{\dot{x}}^2 + \left( \frac{\dddot{x}}{\ddot{z} + g} \right)^2 \sigma_{\dot{z}}^2 \]

\[ + \left( \frac{\dot{z} \dddot{x}}{(\ddot{z} + g)^2} \right)^2 \sigma_{\dot{z}}^2 \]

ie. the standard error in the derived wind (\( \sigma_u \)) is the vector sum of the standard error in the observed horizontal velocity and the standard error in the correction term.