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IMPLICATIONS OF A DIRECT METHOD FOR SOLVING THE HELMHOLTZ-TYPE EQUATION

L. M. Leslie

Commonwealth Meteorology Research Centre, Melbourne

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INTRODUCTION

Many numerical weather prediction models require the equation

$$\nabla^2\phi - q\phi = f(x, y) \quad \ldots . 1$$

to be solved as part of the time-stepping procedure. This includes the barotropic and baroclinic filtered models and, more recently, the semi-implicit schemes. Filtered models which possess the Cressman phase correction (Cressman, 1958) and semi-implicit schemes such as that of McPherson (1971) require eq 1 to be solved with $q > 0$, while filtered models without the phase correction require eq 1 to be solved with $q = 0$ (*ie* Poisson's equation).

Considering the importance of eq 1 it is not surprising that much has been written on the various means of solving it. These methods fall into two broad classes: the iterative methods and the direct ("exact") methods. Perhaps the best known iterative methods are the Liebmann Successive Over-relaxation method (SOR) and the Alternating Direction Implicit method (ADI). The technique of Ogura and Charney (1962), which is referred to as the Dimension Reduction Method (DRM), and the Fourier analysis approach of Hockney (1965) are amongst the most frequently used direct methods. It is a feature of the direct methods that they are far more accurate than iterative methods and are much quicker, particularly when the finite-difference mesh contains a large number of grid points and when good initial "guesses" are not available. So far the iterative methods have been used to solve eq 1 with $q \geqslant 0$ but the direct methods have been applied only to the solution of Poisson's equation.

It appears then, that there is a need for a quick and accurate method for solving eq 1 with $q \leqslant 0$, particularly when a high-resolution semi-implicit model is envisaged, in which case eq 1 would be solved many hundreds of times for a 24-hour forecast. With such a large number of successive solvings of eq 1, very accurate methods must be available to prevent destructive accumulation of error. Recently, two very quick procedures for solving the Poisson equation have been proposed. Ogura (1969) elaborated on the DRM method and solved the tri-diagonal matrix system of equations which the method produces by a Fast Fourier Transform (Cooley and Tukey, 1965), thereby greatly improving its speed. Hirota *et al* (1970) have proposed a generalized "sweep-out" method (GSM) which is faster than the Ogura DRM method and which is considerably more flexible, being applicable to irregularly shaped domains.

Both the DRM and GSM may be extended to solve the Helmholtz-type equation, although the authors made no attempt to do so. In the investigation below, the DRM method has been chosen for extension to the Helmholtz-type equation mainly because the present author invested a considerable amount of effort in optimising a version of it before the GSM was published. Although the GSM represents an advance on the DRM, the basic contention of this study is not affected: there are available direct methods for solving eq 1 which are faster and significantly more accurate than the iterative methods currently still in favour.

METHOD OF SOLUTION

The only case which will be considered here is the solving of eq 1 when ϕ (x, y) takes prescribed values on the perimeter of a rectangular domain, and where $q \geqslant 0$. Suppose the points of a finite difference mesh are labelled (i, j) where i = 0, 1, ..., M, j = 0, 1, ... N. Furthermore suppose $M \geqslant N$. In most limited-area numerical weather prediction grids i will therefore point approximately from west to east. If the five-point approximation to $\nabla^2\phi$ is made, eq 1 may be written as

$$\frac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{h_x^2} + \frac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{h_y^2} - q\phi_{i,j} = f_{i,j} \qquad \ldots 2$$

where i = 1, ...N-1 and j=1, ...M-1. Using a similar development to that of Ogura, eq 2 may be written as

$$\underset{\sim}{\phi}_{i+1},-(2+qh_x^2)\underset{\sim}{\phi}_i + \underset{\sim}{\phi}_{i-1} + \left(\frac{h_x^2}{h_y^2}\right) \underline{\underline{A}}\ \underset{\sim}{\phi}_i = \underset{\sim}{f}_i{}', \qquad \ldots 2a$$

with

$$\underset{\sim}{\phi}_i = \begin{bmatrix} \phi_{i,1} \\ \vdots \\ \vdots \\ o_{i,N-1} \end{bmatrix}, \quad \underset{\sim}{f}_i' = \begin{bmatrix} f_{i,1}-\frac{\phi_{i,0}}{h_y^2} \\ f_{i,2} \\ \vdots \\ f_{i,N-2} \\ f_{i',N-1}-\frac{\phi_{i,1}}{h_y^2} \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} -2 & 1 & 0 & \ldots & \ldots & 0 \\ 1 & -2 & 1 & \ldots & \ldots & 0 \\ & & & & & \\ 0 & \ldots & \ldots & 1 & -2 & 1 \\ 0 & \ldots & \ldots & 0 & 1 & -2 \end{bmatrix}$$

h_x and h_y denoting grid intervals in the x and y directions respectively. Since $\underline{\underline{A}}$ is a real symmetric matrix it follows from a well-known theorem that there exists a diagonalizing matrix $\underline{\underline{T}}$ such that

$$\underline{\underline{T}}\ \underline{\underline{A}}\ \underline{\underline{T}}^{-1} = \underline{\underline{\Lambda}},$$

where $\underline{\underline{\Lambda}}$ is a diagonal matrix. For the present matrix $\underline{\underline{A}}$,

$$\underline{\underline{\tau}} = \begin{bmatrix} \sin \pi/N & & \sin \frac{(N-1)\pi}{N} \\ & & \\ \sin \frac{(N-1)\pi}{N} & & \sin \frac{(N-1)^2 \pi}{N} \end{bmatrix}$$

and

$$\underline{\underline{\Lambda}} = \begin{bmatrix} -4\sin^2 \pi/2N & 0 & 0 \\ 0 & & \\ & & 0 \\ 0 & 0 & -4\sin^2 \frac{(N-1)\pi}{2N} \end{bmatrix}$$

If we define $\underset{\sim}{P}_i$ by $\qquad \underset{\sim}{\phi}_i = \underline{\underline{\tau}}^{-1} \underset{\sim}{P}_i,$ 3

and multiply eq 2 from the left by $\underline{\underline{\tau}}$, then

$$\underset{\sim}{P}_{i+1} - (2+qh_x^2)\underline{\underline{I}} - (\frac{hx}{hy})^2 \underline{\underline{\Lambda}}) \underset{\sim}{P}_i + \underset{\sim}{P}_{i-1} = \underset{\sim}{g}_i, \qquad \text{....4}$$

where $\qquad \underset{\sim}{g}_i = \underline{\underline{\tau}} \, \underset{\sim}{f}_i^{\prime}$ 5

This equation, which differs from equation 2.5 in Ogura (1969) only by the presence of the additional term $qh_x^2 \underline{\underline{1}}$, is a linear tridiagonal system which may be solved by the simple and efficient algorithm devised by Richtmyer (1967). The final solution field $\underset{\sim}{\phi}_i$ may then be obtained from $\underset{\sim}{P}_i$ by observing that $\underline{\underline{\tau}}^{-1} = \frac{2}{N} \underline{\underline{\tau}}$, and therefore from eq 3

$$\underset{\sim}{\phi}_i = \frac{2}{N} \underline{\underline{\tau}} \, \underset{\sim}{P}_i \qquad \text{....6}$$

It is noted that the summations eq 5 and 6 are of the form

$\sum_{\ell=1} \alpha_{i,\ell} \sin \frac{j \ell \pi}{N}$ and were calculated by Ogura using a fast Fourier transform. In this paper however an alternative, and much simpler, method described by Williams (1969) was preferred.

TEST PROBLEM

A comparison was made between a representative of the iterative methods, namely the Sheldon-speeded SOR, and the DRM method by applying each to eq 1 with q equal to a positive constant, and beginning with a "flat" initial guess. The actual value of q used in the test was 20, and was chosen because it gave a value of d^2q comparable with that of the filtered numerical weather prediction models.

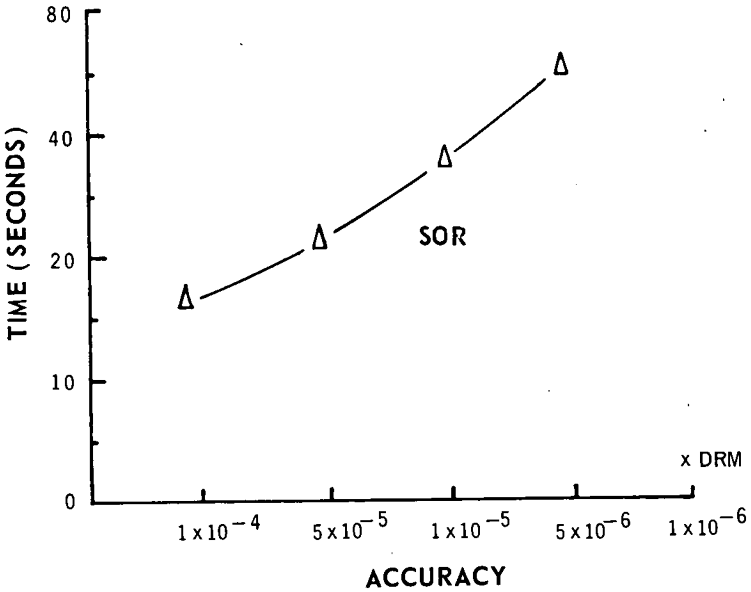

Fig. 1 Plot of accuracy and time required for SOR and DRM methods on the 61 x 61 mesh.

The results of the test are shown in Table 1 and Fig 1, and they illustrate the important advantages of the DRM method - accuracy and speed. In fact, the accuracy of the SOR method could be increased to that of the DRM method only by performing a prohibitive number of iterations.

Another feature of the DRM method is that it runs more quickly the greater the disparity between M and N, a feature of which great advantage may be taken. This property is indicated in Table 1 where the same computation was made but with a 91 x 41 grid instead of the 61 x 61 grid used for most of the calculations. The significant improvement in speed results from the application of the fast fourier transform technique to eq 5 and 6 which cuts down both the j-range and range of fourier summation by a factor of eight when Williams' procedure is used.

In fairness to the SOR method it should be pointed out that if good initial "guesses" are available, if array sizes are not very large, and if moderate accuracy only is required, then the DRM method is no longer overwhelmingly superior. Briefly, for the computations of this paper, the Sheldon-speeded SOR method became economically undesirable if more than about 12 iterations were needed on the 61 x 61 mesh.

Table 1 Comparison of Sheldon-speeded SOR and DRM methods. Flat initial "guess" field.

Mesh / Method	61 x 61		91 x 41	
	SOR	DRM	SOR	DRM
Time (s)	16.9	2.9	16.3	1.6
Max Error	2.9×10^{-4}	4.5×10^{-6}	2.7×10^{-4}	4.3×10^{-6}
No of Iterations	92	-	89	-

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