Generation of synthetic meteorological data sets by linear filtering of white noise

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A method is presented for the generation of meteorological data sets, with specified spatial statistical structures, by the application of appropriately designed linear filters to random white noise. It incorporates inhomogeneity, anisotropy, and geographically variable coupling of height and wind fields. The paper includes details of the basic theory and practical application, together with examples.

Introduction

For purposes such as observational network design, assessment of the likely impact of a new observing system, or the testing of an objective analysis or data assimilation system, it is often convenient to use synthetic sets of meteorological data with properties similar to those of data from the real atmosphere. For example, in the decade preceding the Global Weather Experiment of 1978-79, many so-called observing system simulation experiments (OSSEs) were performed. In these experiments, the synthetic data were usually generated by a dynamical general circulation model of the atmosphere, operating over a hemispheric or global domain.

If one is interested mainly in analysis rather than forecast impact, it may be desirable to generate a large number of independent snapshots of a limited geographical domain, rather than a long time-coherent global four-dimensional data set. Sophisticated general circulation models are less suitable for limited area snapshots, firstly because of their global domain, secondly because the inherently long time-scales in the models (and the real atmosphere) would necessitate long integration periods to achieve a large number of statistically independent data sets, and thirdly because of spatial resolution limitations for mesoscale simulation.

The purpose of this paper is to present a method suitable for generating a large number of statistically independent realisations of meteorological data over a limited area. The method is based upon linear filtering of a series of random numbers (white noise), to induce correlations of a prespecified form. In the statistical literature on time series analysis, the introduction of serial correlation into a random series by application of a filter or moving average has long been known as the Slutsky-Yule effect. In the context of spatial objective analysis of meteorological fields, Thiebaux (1976) pointed out that an appropriate linear filter operating on white noise reproduced certain autoregressive processes whose correlation function corresponded well to observed data. In an application akin to that of the present paper, Pratte and Lee (1979) used two-dimensional filtering of white noise to simulate high resolution (10km) wind data from a proposed satellite-borne Doppler lidar. The present paper generalises and extends the latter work, firstly to allow for generation of multivariate (geopotential and wind) fields which are partially dynamically coupled, and secondly to permit inhomogeneity and anisotropy, both of which are well established properties of the real atmosphere. Features of the method, in comparison with the 'dynamical model-generated data' approach, are its economy and its flexibility to range over a wide parameter space.

The following sections contain details of the basic theory and practical application, and examples of synthetic data sets.

Theory

For simplicity, the theory below is presented for the one-dimensional case, but extensions to additional dimensions are straightforward.

Let $\phi(x)$, $x = 1, 2, \ldots k$, represent a series of random numbers with zero mean and unit variance, spaced at integer intervals along the x-axis. Let $W(\xi)$ represent the weights of a symmetric filter, as a function of integer displacement ξ in the x-direction. For example, the well-known 1-2-1 filter (e.g. Shuman 1957) would correspond to $W(0)=0.5$, $W(-1)=W(1)=0.25$, $W(i)=0$ for $i\neq 0, \pm 1$. If the filter is applied to the random number series, the resultant smoothed series $\phi_s(x)$ is given by

$$\phi_s(x) = \Sigma\, W(\xi)\, \phi(x+\xi) \qquad \ldots 1$$

where the summation is over all ξ. Similarly

$$\phi_s(x+\alpha) = \Sigma\, W(\xi)\, \phi(x+\alpha+\xi) \qquad \ldots 2$$

By definition, the original white noise has the properties

$$\overline{\phi(x)} = 0$$
$$\overline{\phi^2(x)} = 1 \qquad \ldots 3$$
$$\overline{\phi(x)\,\phi(x+i)} = 0, \text{ when } i \neq 0$$

where the overbar indicates the mean over a large number of realisations.

The smoothed series $\phi_s(x)$ will be correlated in the x direction, with the covariance, corresponding to the separation α, defined by

$$R(\alpha) = \overline{\phi_s(x)\,\phi_s(x+\alpha)} \qquad \ldots 4$$

The correlation coefficient $r(\alpha)$ is obtained by combining Eqns 1, 2, 3, 4, using the symmetry property $W(\xi) = W(-\xi)$, and normalising by $R(0)$, so that

$$r(\alpha) = \Sigma\, W(\xi)\, W(\xi+\alpha) / \Sigma\, W(\xi)^2 \qquad \ldots 5$$

Therefore, for a given filter $W(\xi)$, the corresponding correlation function of the smoothed series can be computed directly from Eqn 5.

The problem addressed in this paper is the inverse of the above, namely, given a desired form of correlation coefficient function $r(\alpha)$, to determine the corresponding filter weighting function $W(\xi)$. For their application, Pratte and Lee (1979) solved the two-dimensional analogue of Eqn 5 numerically by trial and error, assuming a sum of exponentials for $W(\xi)$. However, an analytic solution will sometimes be feasible.

Once the appropriate form for $W(\xi)$ has been determined, as many smoothed realisations as desired may be obtained, by application of Eqn 1 to different random number series $\phi(x)$. The smoothed realisations $\phi_s(x)$ will have zero mean and unit variance, and will be correlated in the x-direction according to the covariance function $R(\alpha)$.

The preceding theory forms the basis of the method used in this paper for generating synthetic data sets. Two alternative approaches are worthy of mention. Firstly, it is possible to use the Fourier transform of the covariance function, namely the spectrum function, to generate synthetic data (e.g. Yang and Shapiro 1973; Yang 1974; North and Cahalan 1981). An advantage of using the correlation function, rather than the spectrum function, will become evident in the subsection 'Generalisation for inhomogeneity and anisotropy' of the next section. A second method, mentioned in Ripley (1981), uses factorisation of the correlation matrix to generate correlated series. A scientific subroutine based on this method is available in IMSL (1982). As both Franke (1985), and the author, have found with this approach, conditioning problems severely restrict matrix size when realistic correlation functions and data spacings are used. However, Franke (1986) has also shown that the ill-conditioning problem may be alleviated by adding a small term to the diagonal of the correlation matrix, thereby allowing a larger matrix size to be used.

Application and examples

Overview

The practical application to be described was the generation of two-dimensional data sets of synthetic synoptic observations of height and wind on an isobaric surface. The corresponding height and wind fields were subject to a quasi-geostrophic constraint, the strength of which varied with geographic location.

The overall task was divided into the following steps:

1. The appropriate filter was designed.
2. Independent realisations of white noise were generated on a two-dimensional grid, and normalised fields were then synthesised by application of the filter.
3. The normalised fields were coupled, as appropriate, to simulate normalised height, stream function and velocity potential.
4. The fields were denormalised, using appropriate means and standard deviations, to produce fields of height and wind components.
5. Observations were interpolated from the grid-point fields, and realistic observational errors added.

The steps (1) to (5) are detailed in the following subsections.

The two-dimensional isotropic filter corresponding to a Gaussian correlation function

Because most observed spatial correlations are close to zero at separations greater than a few thousand kilometres, a tangent plane approximation to a limited area of the earth's surface appears reasonable and is used in the following two-dimensional discussion.

The filter design problem may be addressed analytically by considering the two-dimensional integral equation corresponding to Eqn 5. A filter $W(\xi,\eta)$ is required to satisfy

$$\iint_{-\infty}^{\infty} W(\xi,\eta)W(\xi+x,\, \eta+y)\, d\xi d\eta \Big/ \iint_{-\infty}^{\infty} W(\xi,\eta)^2\, d\xi d\eta$$
$$= r(x,y) \qquad \ldots 6$$

where $r(x,y)$ is the desired correlation function of two-dimensional separation (x,y). When the desired correlation function is isotropic, the corresponding filter will be circularly symmetric, so that if

$$r(x,y) = r(s)\ ;\ s^2 = x^2 + y^2$$

then $W(\xi,\eta) = W(\gamma)\ ;\ \gamma^2 = \xi^2 + \eta^2$

The examples presented in this paper utilise the Gaussian correlation function, which in isotropic form is

$$r(s) = \exp(-0.5\, s^2/L^2) \qquad \ldots 7$$

where L is a length scale parameter. While there are certainly other functions (Julian and Thiebaux 1975; Thiebaux 1976; Hollingsworth and Lonnberg 1986) which fit observed correlations better than the Gaussian, the latter was chosen here for demonstration purposes because of its familiarity and widespread use, and because the corresponding filter has a simple analytic form.

It is shown in the Appendix that when r(x,y) in Eqn 6 is the Gaussian function Eqn 7, then the solution of Eqn 6 for $W(\xi,\eta)$ is

$$W(\xi,\eta) = W(\gamma) = \exp(-\gamma^2/L^2) \qquad \ldots 8$$

In practice, the filter must be applied in discrete form at grid-points, with the infinite integral in Eqn 6 replaced by a finite summation. Under these circumstances, the resultant correlation coefficients will no longer be exactly Gaussian. The exact values of correlation coefficient corresponding to the discretised and truncated filter can be calculated from the two-dimensional analogue of Eqn 5. Their departures from the desired Gaussian values are only small; for a grid length of 0.25L, and truncation at 2L, the root mean square departure is 0.00134.

Generalisation for inhomogeneity and anisotropy

The discussion in the preceding subsection is now generalised for possible inhomogeneity and anisotropy, both of which are well established characteristics of some geographical areas. For example, in the Australian region, length scales of height are characteristically greater in the tropics than in mid-latitudes, and systematic anisotropy is present with the major axis orientation changing between tropics and mid-latitudes (Seaman 1982; Buell and Seaman 1983).

Because the filter (Eqn 8) is formulated in terms of length scale L, it is straightforward to incorporate inhomogeneity by allowing L to vary with the location of the grid-point at which the filter is centred. Similarly anisotropy is incorporated by taking into account the variation of length scale with direction. The latter is accomplished by using a scaled two-point separation s_*, as in Seaman (1982), defined by

$$s_*^2 = s^2\,((\cos^2(\Theta-\lambda)/E^2) + (E^2 \sin^2(\Theta-\lambda))) \qquad \ldots 9$$

where E is an ellipticity parameter, λ is the orientation (clockwise from north) of the major axis of elliptical correlation coefficient contours, and S and Θ are respectively the two-point separation, and the orientation of the line joining the two points (see Fig. 1). The filter corresponding to the Gaussian correlation function in the inhomogeneous, anisotropic case is therefore

$$W(s_*) = \exp(-s_*^2/L^2) \qquad \ldots 10$$

where L varies according to the point at which the filter is centred.

The correlation coefficient between any pair of points, resulting from these generalisations, may be computed from Eqns 1 to 5, taking into account the spatial variation of weights.

Normalised two-dimensional inhomogeneous anisotropic realisations of grid-point fields can therefore be generated by applying Eqn 10 to white noise realisations which are available from standard scientific subroutines. Figure 2 shows samples of normalised realisations using the four possible combinations of (in)homogeneity and (an)isotropy.

The preceding formulation highlights the convenience of working with the correlation function and linear filtering, rather than with the spectrum function. With the intrinsically global character of the latter, it would be less easy to incorporate realistic inhomogeneities and variations in anisotropy within the domain.

Partially coupled height and wind fields

Normalised fields of the type shown in Fig. 2 can be considered as fields of standard deviations from the mean, for any meteorological element. In order to generate realistic concurrent fields of height (z) and wind, it is necessary to take into account relevant multivariate relations between those two fields.

Following the general approach of Lorenc (1981) and Daley (1985), the total wind $\vec{V}$ is considered as the sum of its rotational and divergent components according to Helmholtz's theorem

$$\vec{V} = \vec{k} \times \nabla\psi + \nabla \cdot \chi$$

where ψ and χ are stream function and velocity potential, and $\vec{k}$ is the vertical unit vector. It is necessary to prespecify, at each grid-point, the cross-correlation coefficient $r_{z\psi}$, $r_{\psi\chi}$ and $r_{z\chi}$ between z, ψ and χ, and a parameter D^2 which is the ratio of the divergent wind variance to the total wind variance.

Fig. 1 Illustrating the parameters S,E,Θ and λ used in the representation of anisotropic correlations.

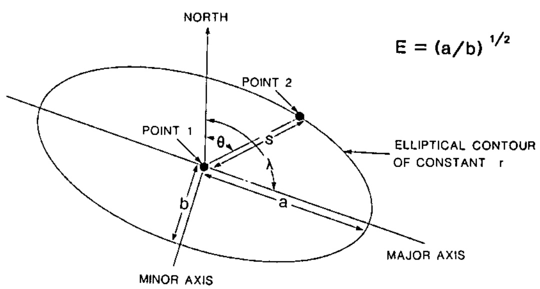


Fig. 2(a) A realisation of a homogeneous isotropic normalised field corresponding to a Gaussian correlation function, L_G = 1000 km. Units are standard deviations times 10.

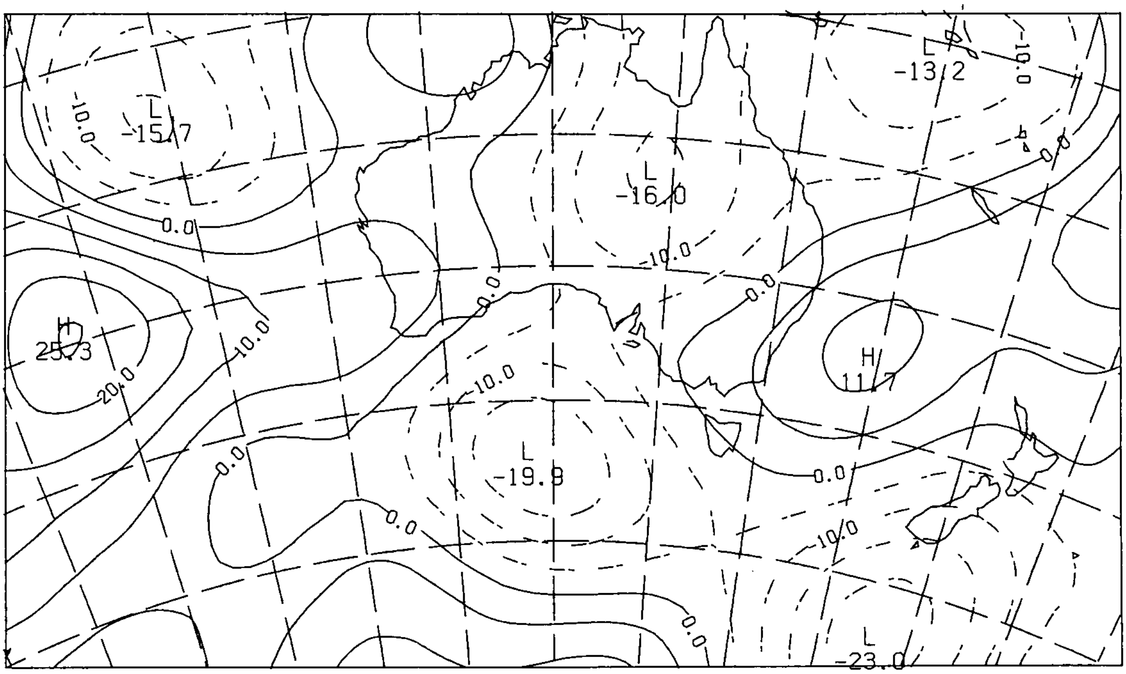


Fig. 2(b) As in 2(a), except inhomogeneous with L_G ranging from 600 km at 60 south to 1200 km at the equator.

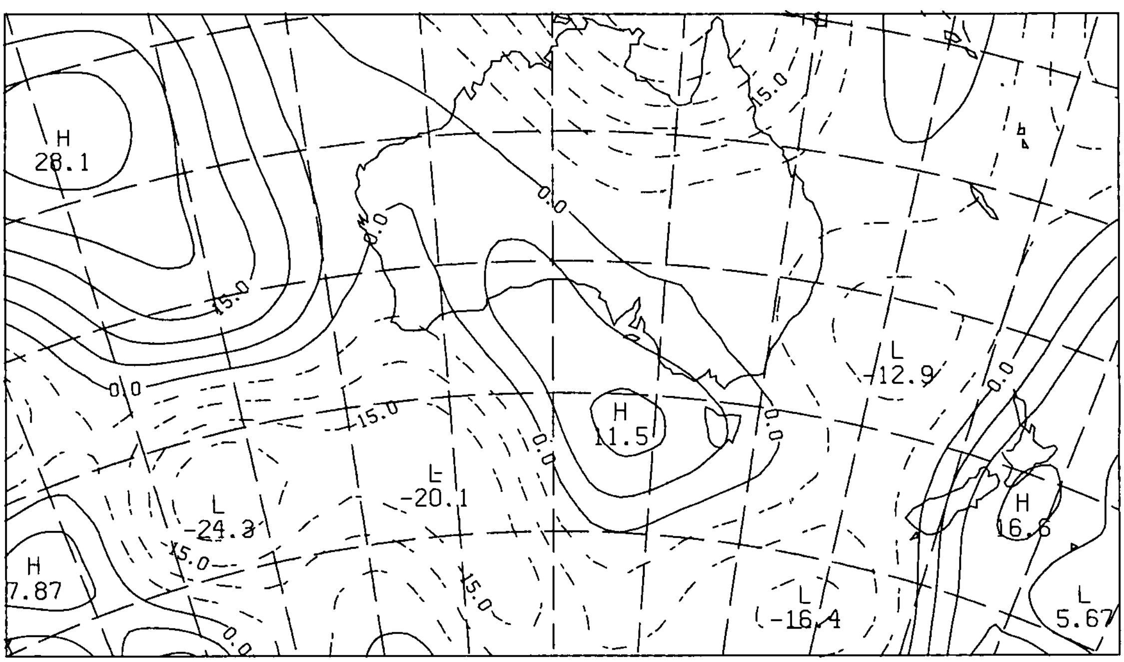


Fig. 2(c) As in 2(a), except anisotropic with $E^2 = 1.5$, $\lambda = 120°$.

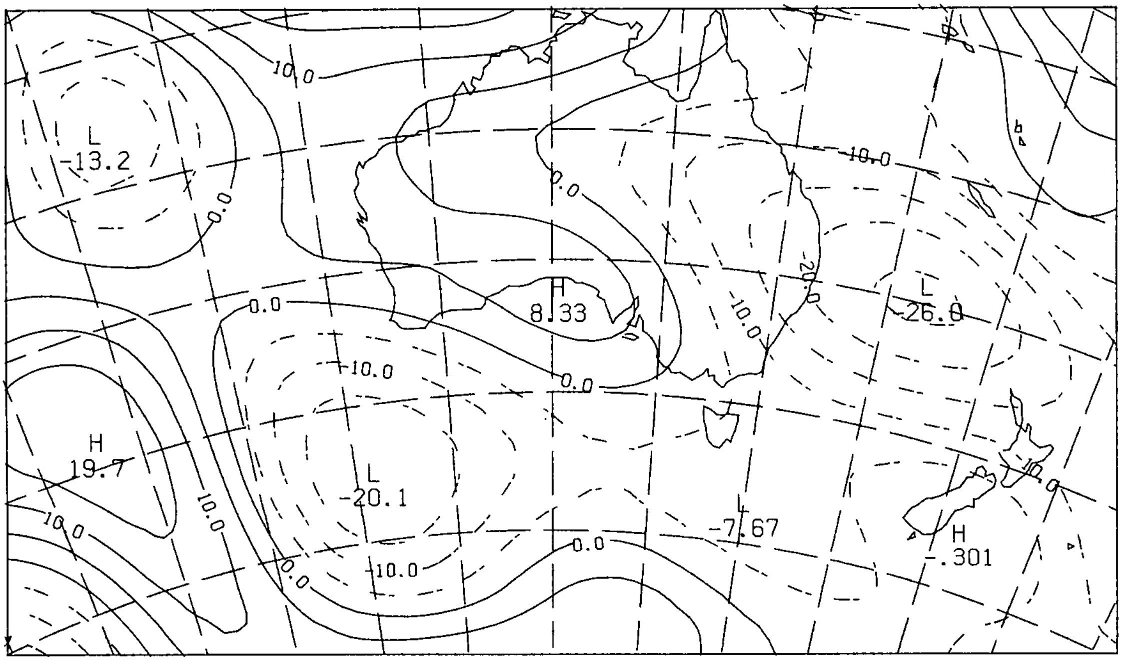
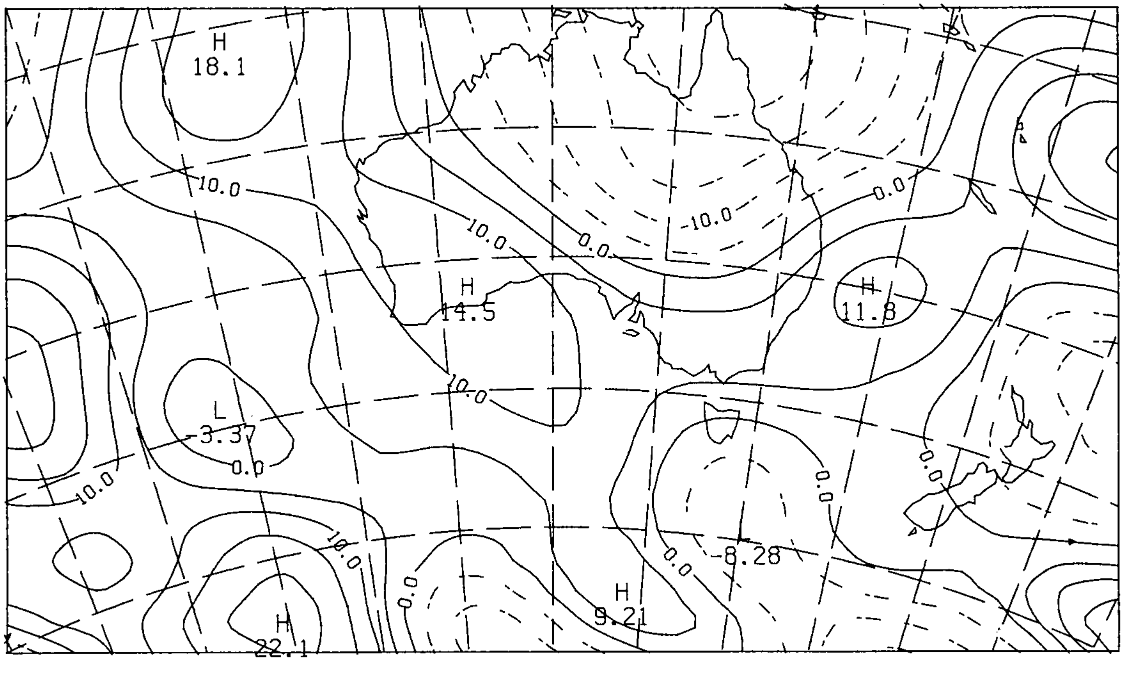


Fig. 2(d) As in 2(a), except inhomogeneous and anisotropic.



These four parameters in general will vary with latitude; for example, $r_{z\psi}$ will be much higher in mid-latitudes than in the tropics, and vice versa for D^2.

Having specified fields of $r_{z\psi}$, $r_{\psi\chi}$, $r_{z\chi}$ and D^2, fields of z_N, ψ and χ may be generated from normalised field realisations a, b and c as follows.

$$\begin{aligned}
\text{Let } z_N &= a \\
\psi &= (r_{z\psi}\, a + (1 - r_{z\psi}^2)^{1/2}\, b)\,(1 - D^2)^{1/2} \\
\chi &= (\alpha a + \beta b + \gamma c)\, D \\
\text{where } \alpha &= r_{z\chi} \\
\beta &= (r_{\psi\chi} - r_{z\chi}\, r_{z\psi}) / (1 - r_{z\psi}^2)^{1/2} \\
\gamma &= (1 - \alpha^2 - \beta^2)^{1/2} \qquad \ldots 11
\end{aligned}$$

It can be confirmed by evaluating $\overline{z_N\psi}$, $\overline{\psi\chi}$, $\overline{z_N\chi}$, and the corresponding variances, from Eqn 11, that the fields z_N, ψ and χ generated as above have the desired cross-correlations.

Fields of wind components u and v can now be computed by finite differencing of ψ and χ. Under homogeneous and isotropic conditions, and neglecting truncation errors, such wind components would have variances of $1/L^2$ (Buell 1957), and under those conditions could be renormalised to unit variance by multiplying by L. Truncation error could also be taken into account, if so desired, by using the variance of the finite difference representation rather than the variance of the continuous derivative. However, in the present application, the wind components u and v have been renormalised to approximately unit variance simply by using the local value of L (or in the anisotropic case, the corresponding anisotropic length scales). Denote the renormalised components by u_N and v_N.

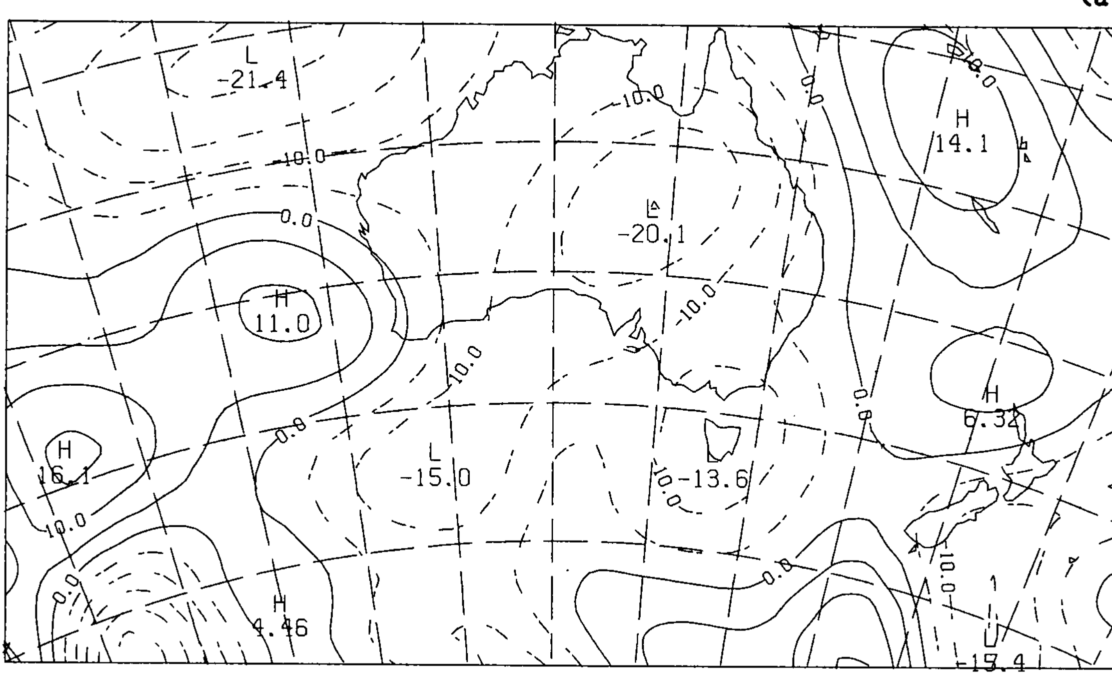
In order to denormalise z_N, u_N and v_N into dimensional units, it is necessary to multiply by the appropriate standard deviation and to add the climatological mean. Climatological fields for the southern hemisphere January, based on years 1972 to 1981, from Le Marshall et al. (1985), were used in the examples shown here, although for high resolution simulations a more detailed climatology would be desirable.

The relation between the variances of height and wind is important for geostrophic coupling. Buell (1954) shows that for geostrophic flow, and neglecting the horizontal gradient of height variance, the ratio of wind component variance to height variance is $(g/Lf)^2$. This additional constraint was imposed upon the climatological variances, by adjusting them to the appropriate ratio, and regressing towards the adjusted values according to $r_{z\psi}$. Such a procedure ensures close to geostrophic coupling when $D = 0$ and $r_{z\psi} = 1$.

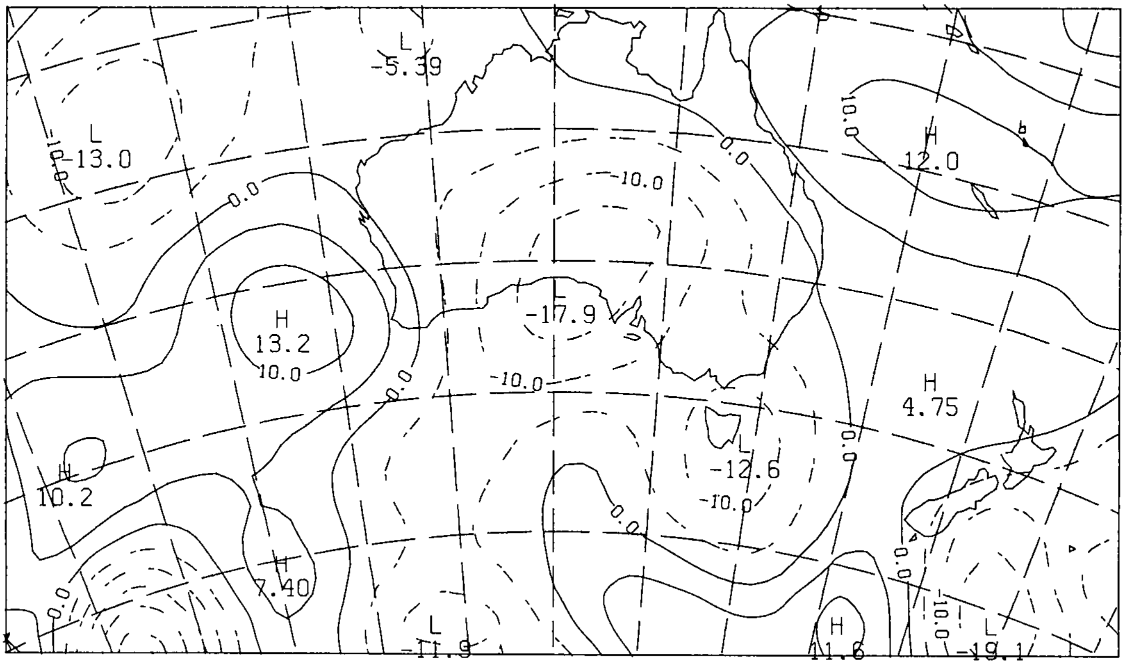
Examples of normalised height, stream function and velocity potential fields, of the corresponding denormalised 850 mb height and wind fields, and of 850 mb streamlines are shown in Fig. 3. The values of the relevant parameters appear on the caption. Note the progressive decoupling with latitude of the height and wind fields, the greater divergent component in

Figs 3(a) to (c) Coupled fields of (a) height z_N, (b) stream function ψ, (c) velocity potential χ, normalised as described in the text. The parameter D ranges from 0.2 (poleward of 30 degrees) to 0.7 (equatorward of 18), $r_{z\psi}$ ranges from 0.95 (poleward of 30) to 0.0 (equator), $r_{\psi\chi} = 0.99$, $r_{z\chi} = r_{z\psi} r_{\psi\chi}$. Other parameters as in 2(b). Contour values have been multiplied by 10.

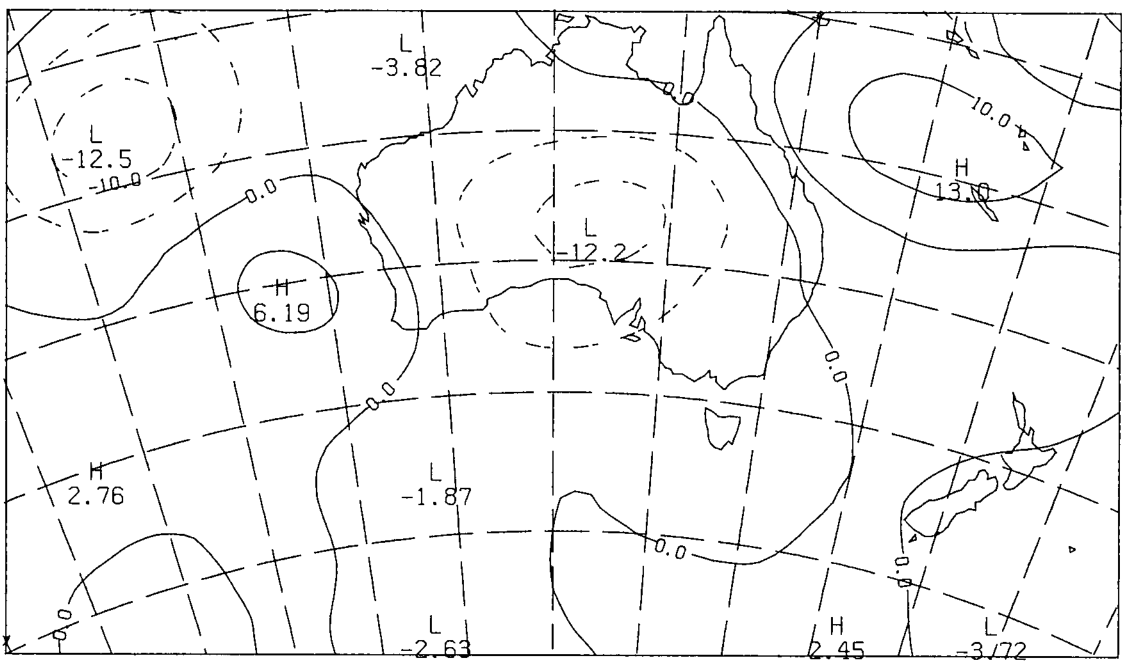
(a)



(b)



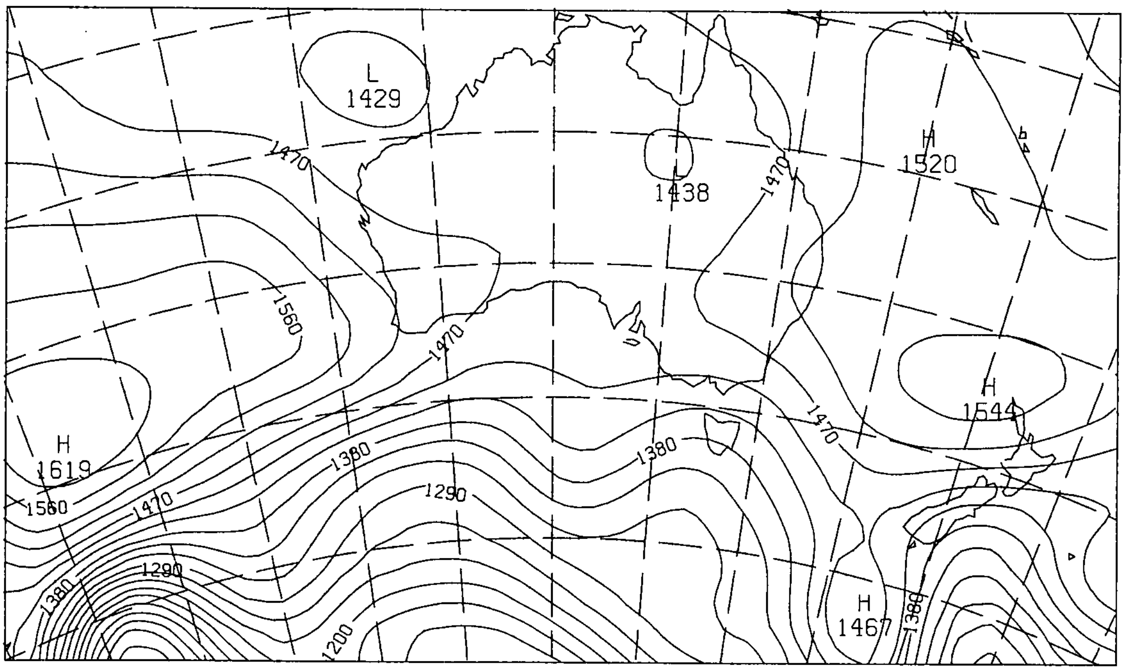
(c)



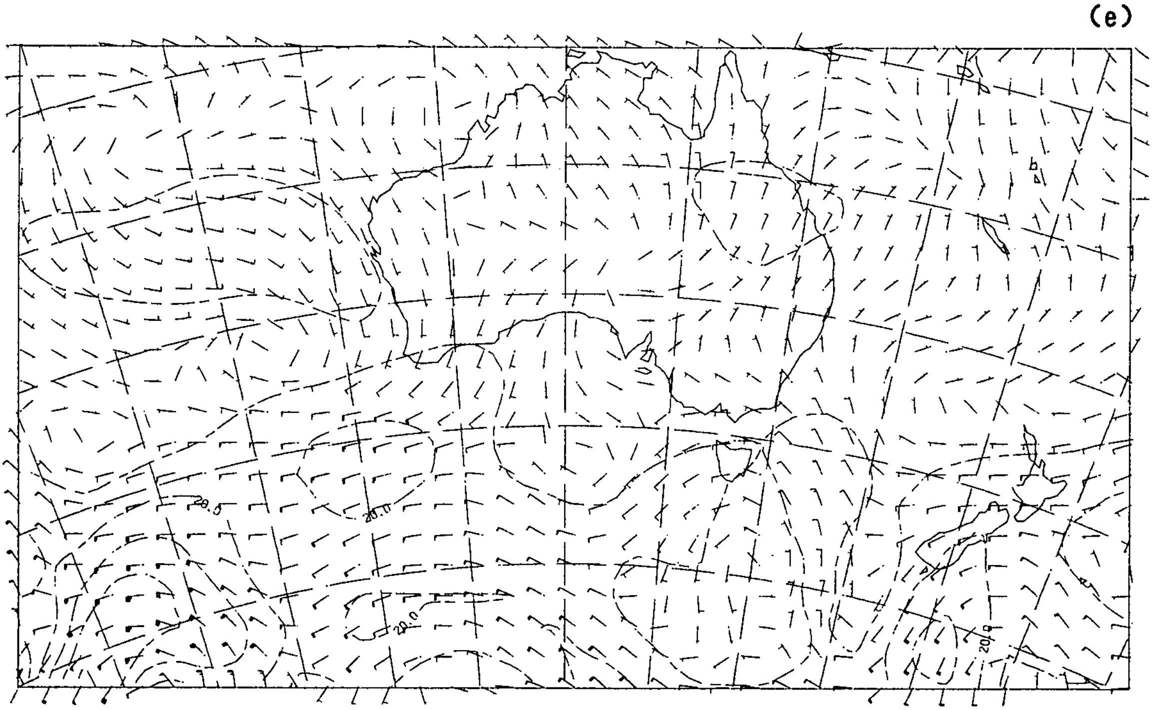
the tropics, and the inflows and outflows associated with cyclonic and anticyclonic circulations. The parameters were deliberately chosen to illustrate the effects just mentioned and (especially in the case of $r_{\psi x}$) have no strong foundation in observational statistics.

Figs 3(d) to (f) **Denormalised fields of (d) 850 hPa height, (e) 850 hPa wind, and (f) 850 hPa streamlines corresponding to Figs 3 (a) to (c), and using January climatological means and standard deviations. Units are metres and m s^{-1}.**

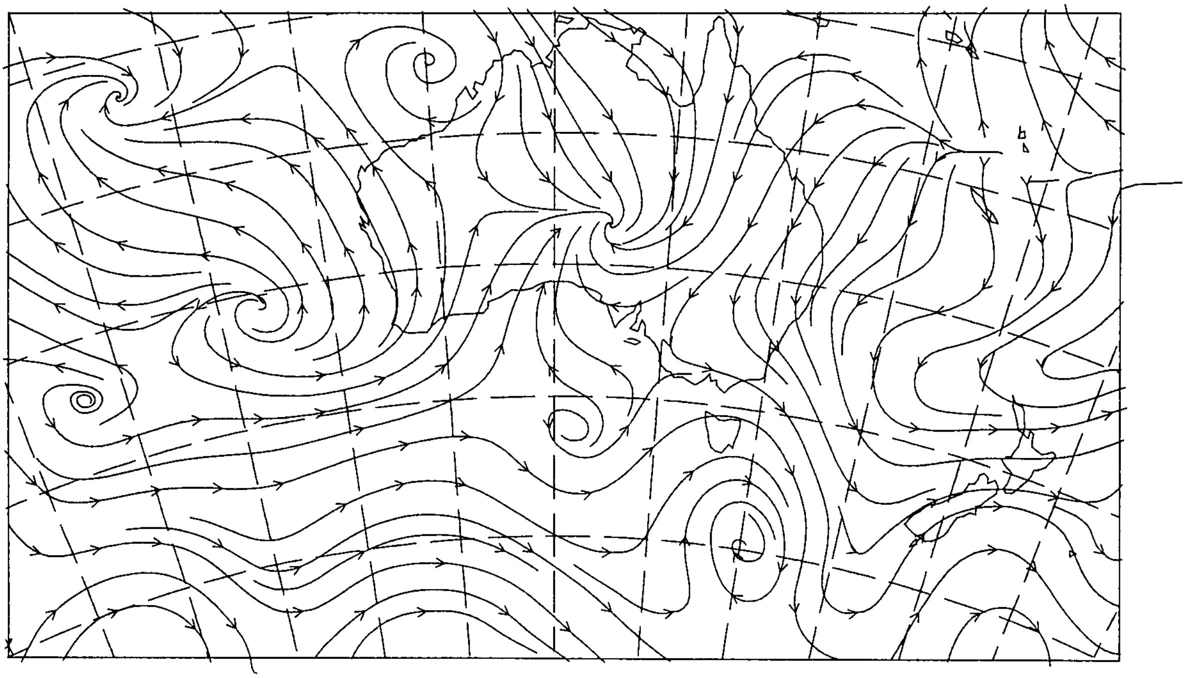
(d)



(e)



(f)



Generation of synthetic observations

The most straightforward way to generate irregularly spaced synthetic observations, is to interpolate them directly from the corresponding grid-point field, and to add realistic observational errors. Because the interpolation method itself will have its own characteristic transfer function, the spectrum (or correlation function) of the synthetic observations will not correspond precisely to that of the grid-point field. However, provided that the computational grid length is much less than the characteristic length scale, the uncertainty introduced by interpolation is probably negligible, and in any case has been neglected in this implementation. Such uncertainty could, if desired, be avoided entirely by using a sufficiently fine grid, and taking a subset of that grid as 'observations'.

In the present application, random observational point locations of a prescribed average spatial density, were generated using a standard random number subroutine. Bicubic interpolation was used to interpolate 'true' values at observation locations, then normally distributed 'errors' of a prescribed standard deviation were added to the 'true' values. It would, of course, be possible to simulate spatially correlated observational errors, such as occur with satellite-based temperature soundings, by interpolating from a spatially correlated 'observational error' field generated using the same approach as in subsections 'The two-dimensional isotropic filter...' and 'Generalisation for inhomogeneity and anisotropy'.

Figure 4 shows an example of concurrent height and wind observations, generated from the fields of Fig. 3. The error parameters used appear on the caption.

Concluding remarks

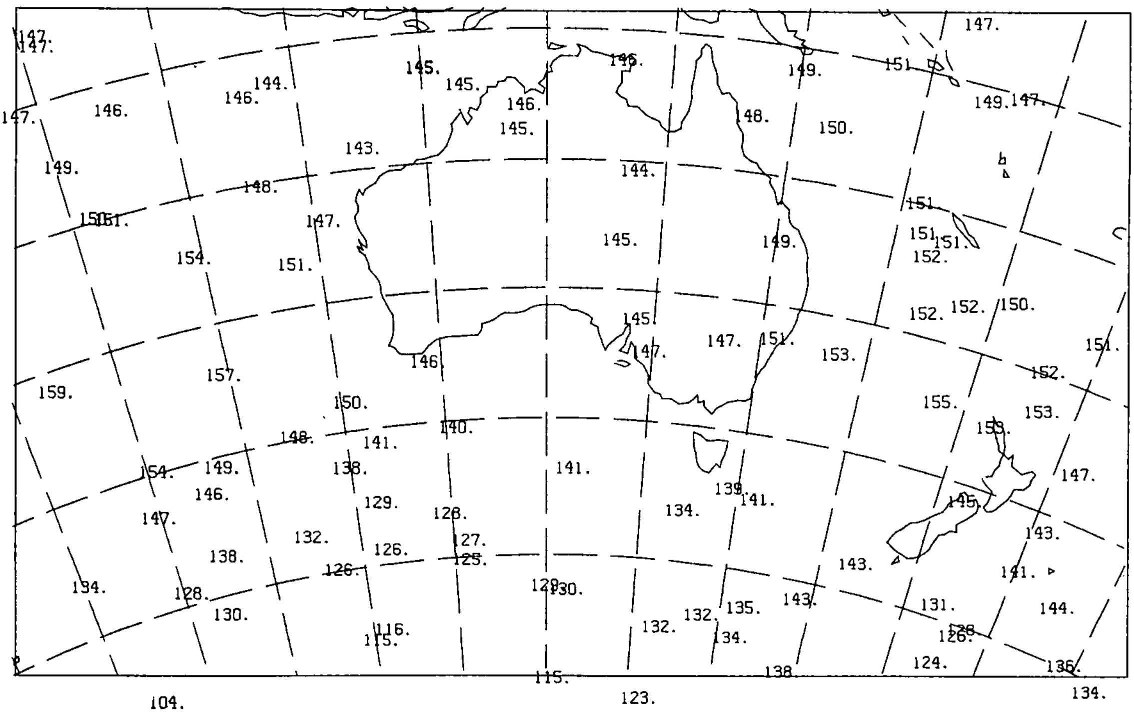
The essential ingredients of the methodology presented above are knowledge of the relevant statistical characteristics of the data sets to be simulated, the use of an appropriate linear filter on white noise to induce the desired spatial structure, and the use of the quasi-geostrophic assumption as a space-variable constraint. The method is probably best described as a mainly statistical simulation, with only rudimentary dynamics.

The advantages of the approach are its ability to produce a large number of independent snapshots of the (simulated) atmosphere, its flexibility with respect to scale, domain, and inhomogeneity and anisotropy within the domain, and when compared with traditional dynamical model simulations, its economical requirements for computer storage and computing time. Possible applications are in observing system simulation, in the design of observational networks, and in the development and comparison of objective analysis algorithms.

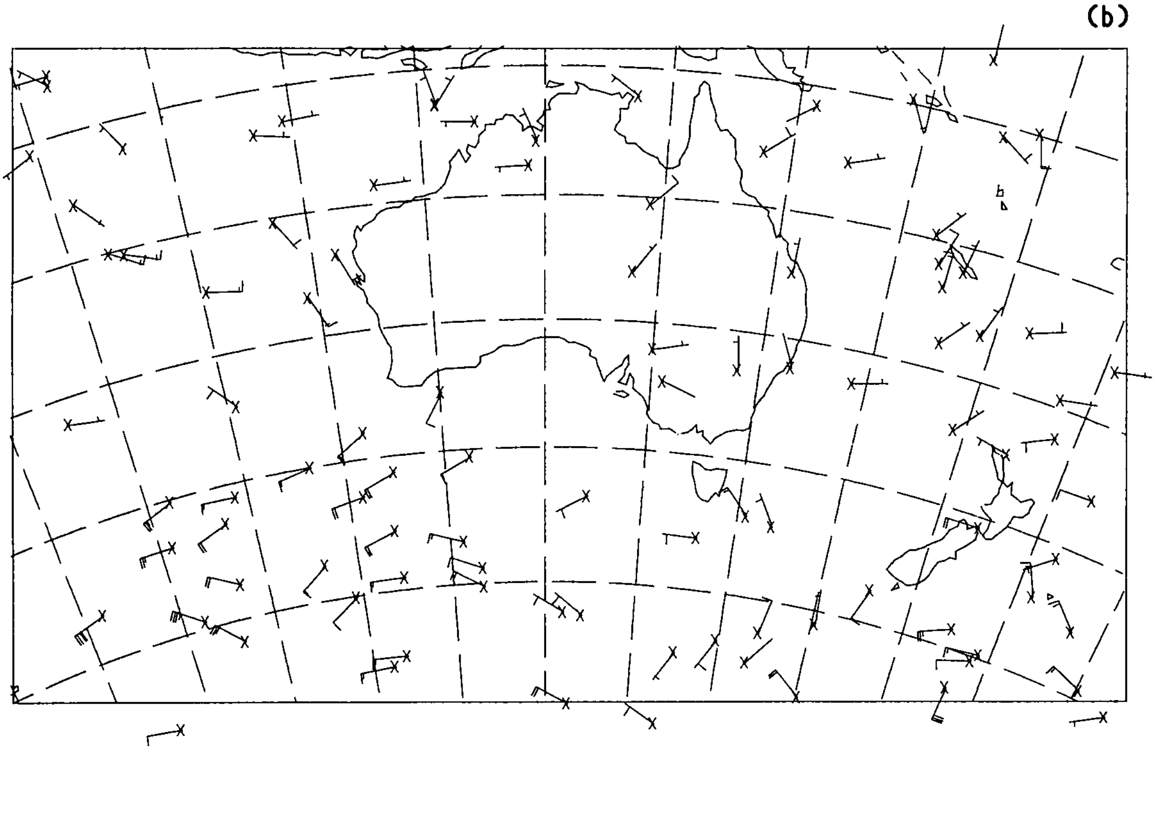
While the present implementation has utilised normalised increments from climatological mean fields, it would also be possible to generate increments from typical background ('first guess') fields in a data

Fig. 4 Height and wind observations generated from fields in Figs 3(d) to (e). Root mean square observation errors of 5 m and 3 m s^{-1} were added. The observations are randomly distributed with an average spatial density of about one observation per (700 × 700) km.

(a)



(b)



assimilation scheme. The latter procedure would have the additional advantage of incorporating realistic dynamics through the background field. Another possible application may be to generate synthetic data for particular phenomena (e.g. tropical cyclones, fronts, etc.) with known or presumed statistical structures.

While the filter corresponding to the Gaussian correlation function can be determined analytically, this may not always be possible. In the case of a second order autoregressive (SOAR) correlation function, Thiebaux (1976) has shown that an analytic solution exists when the two-dimensional correlation function is separable and can be expressed as a product of orthogonal one-dimensional SOAR correlation functions. However in the absence of such a factorisation, or for functions other than those mentioned, the situation is less clear. If no analytic solution is available, the numerical approach of Pratte and Lee (1979) may have to be used. Work currently underway suggests that the latter approach provides a close approximation to a two-dimensional non-separable SOAR correlation function.

The most obvious extension of the present work is to three or four dimensions. Here too the question of separability may be important, because as Pratte and Lee suggest, the multidimensional problem is simplified if horizontal, vertical and temporal filters can be applied sequentially. That simplification implies that the corresponding correlation function may be similarly factorised, which is an assumption also made in many practical objective analysis schemes.

Acknowledgments

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References

Apelblat, A. 1983. *Tables of definite and infinite integrals.* Elsevier, 457pp.

Buell, C.E. 1954. Some relations among atmospheric statistics. *J. Met., 11,* 238-44.

Buell, C. E. 1957. The correlation between wind and height on an isobaric surface. *J. Met., 15,* 309-16.

Buell, C. E. and Seaman, R. S. 1983. The 'scissors' effect; anisotropic and ageostrophic influences on wind correlation coefficients. *Aust. Met. Mag., 31,* 77-83.

Daley, R. 1985. The analysis of synoptic scale divergence by a statistical interpolation procedure. *Mon. Weath. Rev., 113,* 1066-79.

Franke, R. 1985. Sources of error in objective analysis. *Mon. Weath. Rev., 113,* 260-70.

Franke, R. 1986. Covariance functions for statistical interpolation. *Tech. Rep. NPS-53-86-007.* Naval Postgraduate School, Monterey, USA.

Hollingsworth, A. and Lonnberg, P. 1986. The statistical structure of short range forecast errors as determined from radiosonde data. Part 1: The wind field. *Tellus, 38A,* 111-36.

IMSL, Inc. 1982. *IMSL Library Reference Manual,* Edition 9. IMSL, Houston, USA.

Julian, P. R. and Thiebaux, H. J. 1975. On some properties of correlation functions used in optimum interpolation schemes. *Mon. Weath. Rev, 103,* 605-16.

Le Marshall, J. F., Kelly, G. A. M. and Karoly, D. J. 1985. An atmospheric climatology of the southern hemisphere based upon ten years of daily numerical analyses (1972-82): 1 Overview. *Aust. Met. Mag., 33,* 65-85.

Lorenc, A. 1981. A global three-dimensional multivariate statistical interpolation scheme. *Mon. Weath. Rev., 109,* 701-21.

North, G. R. and Cahalan, R. F. 1981. Predictability in a solvable stochastic climate model. *J. Atmos. Sci., 38,* 504-13.

Pratte, J. F. and Lee, R. W. 1979. A short method of generating meteorological fields for simulation studies. *Jnl appl. Met., 18,* 1670-3.

Ripley, D. B. 1981. *Spatial statistics.* John Wiley, 252 pp.

Seaman, R. S. 1982. A systematic description of the spatial variability of geopotential and temperature in the Australian region. *Aust. Met. Mag., 30,* 133-41.

Shuman, F. G. 1957. Numerical methods in weather prediction. II Smoothing and filtering. *Mon. Weath. Rev., 85,* 357-69.

Thiebaux, H. J. 1976. Anisotropic correlation functions for objective analysis. *Mon. Weath. Rev., 104,* 994-1002.

Yang, C. 1974. A controlled experiment with one-dimensional interpolation. *Jnl appl. Met., 13,* 625-36.

Yang, C. and Shapiro, R. 1973. The effects of the observational system and the method of interpolation on the computation of spectra. *J. Atmos. Sci, 30,* 530-6.

Appendix

The filter corresponding to the Gaussian correlation function

Denote by R(x,y) the two-point covariance corresponding to a separation (x,y) in a field of white noise filtered by $W(\xi,\eta)$. From Eqns 4, 5 and 6 in the main text, the covariance R is related to the filter W by

$$R(x,y) = \iint_{-\infty}^{\infty} W(\xi,\eta)\, W(\xi+x,\eta+y)\, d\xi\, d\eta \qquad \ldots A1$$

Consider now the specific circular symmetric filter

$$W(\gamma^2) = W(\xi^2+\eta^2) = \exp\left(-(\xi^2+\eta^2)\right) \qquad \ldots A2$$

Substituting Eqn A2 in the right hand side of Eqn A1, and rearranging terms,

$$R(x,y) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \exp\left(-2y^2-2y\eta-\eta^2\right) d\eta \right] \exp\left(-2x^2-2\xi x-\xi^2\right) d\xi \qquad \ldots A3$$

The integral inside the square brackets is of the form

$$I = \int_{-\infty}^{\infty} \exp(-at^2-bt-c)dt \qquad \ldots A4$$

with $a>0$. This is an integral which has an exact value (e.g. Apelblat 1983) given by

$$I = (\pi/2)^{1/2} \exp\left(b^2/4a-c\right) \qquad \ldots A5$$

Therefore, from Eqns A3, A4 and A5,

$$R(x,y) = (\pi/2)\exp(-0.5(x^2+y^2)) = (\pi/2)\exp(-0.5s^2)$$

where $s^2(=x^2+y^2)$ is the square of the two-point separation. The correlation function is given by

$$r = R(x,y)/R(0,0)$$
$$= \exp(-0.5s^2).$$

If all preceding lengths are now scaled by the Gaussian length scale parameter L, it follows that the circular symmetric filter

$$W = \exp(-\gamma^2/L^2)$$

corresponds to the Gaussian correlation function

$$r = \exp(-0.5s^2/L^2)$$

Note that the tangent plane approximation was essential for the factorisation at Eqn A3.