Plumes, entrainment and B.R. Morton

Peter G. Baines
Department of Infrastructure Engineering, University of Melbourne

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This article presents a modern summary of the development of dynamical models of turbulent plumes for use in modelling a variety of natural phenomena in the atmosphere and ocean, and engineering applications. Bruce Morton played a pioneering role in this work, and this article is a tribute to his memory as a friend, scientist and colleague.

Introduction

Bruce Morton's early research career was centred on the dynamics of plumes, which is the focus of most of this article. As described below, the process of entrainment plays an important role in plume dynamics, at least in the mathematical models he studied. And in his teachings he made a point of emphasising the importance of vorticity in understanding fluid flow. 'Vorticity *is* the flow field', a truism he was fond of saying, which of course is true except for the irrotational part of the flow, governed by (for incompressible fluids) Laplace's equation (and hence 'easily understood'). Bruce's views on vorticity were expressed in his 1984 paper 'The generation and decay of vorticity', which has had considerable impact (134 citations to date), and gives a very readable account of where vorticity comes from, and where it goes. But the topic does not loom large in his work on plumes, and for those interested in more details I recommend the mathematically rigorous monograph *Vortex Dynamics* by Phillip Saffman.

Plumes and entrainment

Unlike most scientists, Bruce Morton's best-known and most celebrated paper is the first one he wrote: 'Turbulent gravitational convection from maintained and instantaneous sources', with Stewart Turner and Sir Geoffrey Taylor, *Proc. Roy. Soc., 234,* 1–23, in 1956 (hereafter denoted MTT). Bruce arrived in Cambridge in 1952, and the theoretical problem of turbulent convection from isolated sources was suggested by his supervisor, George Batchelor. Stewart Turner arrived a year later, and as a designated experimenter was assigned as a student to Alan Townsend (all of them Australians, one might add, though Bruce was perhaps a New Zealander). During a year of absence by Townsend, Stewart was persuaded to do some experiments relating to the convection model, which are described in MTT, so that the latter contains material from both their PhD theses. G.I. Taylor wrote little of MTT but, as he explains in a short appendix, after its completion he realised that the theoretical content was very similar to some work that he did in wartime, had presented at a meeting in New Zealand in 1949, but had not yet published. The appendix does not state, but the wartime application included a dynamic breakwater employing air bubble plumes for the 'mulberry' harbours that were constructed following the D-Day landings in Normandy in June, 1944 (Turner 1973).

At a recent count, MTT had over 990 citations, at an increasing rate of over 50 per year. The paper describes the steady flow (a plume) from a constant source of buoyant fluid, in both uniform and stably stratified environments, and also the rise of fluid (a thermal) from an instantaneous source, under the same conditions. These flows are turbulent, and only the bulk quantities—the upwards motion of mass, momentum and buoyancy, are described by the model. For maintained plumes, the relevant variables in standard notation are: the mean upward velocity $\overline{w}$, the plume radius b, the buoyancy (or reduced gravity) g', and the buoyancy (or Brunt-Väisälä) frequency of the environment, $N(z)$, where z is the vertical coordinate. A central feature of the model (for which G.I. Taylor is credited) is the entrainment assumption, which assumes that the rising turbulent motion in the plume causes an inflow of environmental fluid at a rate that is proportional to the mean upward velocity, $\overline{w}$. If this inflow velocity has a proportionality constant α, the mean equations for the upward flux of mass ($Q = \pi b^2 \overline{w}$), momentum ($M = \pi b^2 \overline{w}^2$) and buoyancy ($F = \pi b^2 \overline{w} g'$) have the form (MTT, Turner 1973)

Corresponding author address: Peter Baines, Infrastructure Engineering, University of Melbourne VIC 3010, Australia. Email: p.baines@unimelb.edu.au.

$$\frac{d\left(b^2\overline{w}\right)}{dz} = 2\alpha b\overline{w},$$
$$\frac{d\left(b^2\overline{w}^2\right)}{dz} = b^2 g', \qquad \text{...(1)}$$
$$\frac{d\left(b^2\overline{w}g'\right)}{dz} = -b^2\overline{w}N^2(z).$$

Here $g' = g\Delta\rho/\rho_0$, where g denotes gravity, ρ_0 is a reference density and $\Delta\rho$ denotes the difference between the mean density within the plume and the density $\rho_e(z)$ of the environment at the same level, and $N^2(z) = -\frac{g}{\rho_0}\frac{d\rho_e}{dz}$.

If the environment is not stratified (so that $N(z) = 0$), the total buoyancy flux F is uniform with height, and the above equations have an analytic solution of the form

$$b = \frac{6}{5}\alpha z,\ \overline{w} = \frac{5}{6\alpha}\left(\frac{9\alpha F}{10\pi}\right)^{1/3}\frac{1}{z^{1/3}},\ g' = \frac{5F}{6\pi\alpha}\left(\frac{9\alpha F}{10\pi}\right)^{-1/3}\frac{1}{z^{5/3}}, \qquad \text{...(2)}$$

where the source is at a point at level $z = 0$. This shows that the width of the plume increases linearly with height, and that the velocity and buoyancy continually decrease as the height increases without limit. If the environment is stratified *($N(z) > 0$)*, solutions must be obtained numerically, up to a level where, first, the buoyancy vanishes. Above this level of zero buoyancy the fluid is carried upward by the momentum of the plume to a maximum height where the velocity $\overline{w}$ vanishes. The fluid then falls back to an equilibrium level where it spreads laterally as a neutrally buoyant intrusion. Such behaviour is known as a fountain, rather than a plume, and requires a different, modified model. None-the-less, the maximum height (where $\overline{w} = 0$) given by the MTT model is proportional to where $\left(F_0/N^3\right)^{1/4}$ is the initial buoyancy (or heat) flux, a relationship that is in agreement with the laboratory experiments, and also with a wide variety of plumes in the environment and volcanic eruptions covering a large range of intensities (Sparks et al. 1997). The paper then goes on to derive corresponding equations for instantaneous sources, and makes similar comparisons between the motion of thermals derived from these equations with observations (notably, maximum rise distance) of laboratory thermals. It then finishes with a discussion of applications to atmospheric sources. MTT makes no allowance for the effect of background crossflow relative to the source (which would give bent-over plumes), but otherwise it covers all the main features of buoyant convection from maintained and instantaneous sources, and is the main basis of many studies and applications today.

MTT vs. Priestley and Ball

Before the submission of MTT, another paper appeared on the same topic, by C.H.B. ('Bill') Priestley and Keith Ball, of CSIRO Aspendale (Priestley and Ball (1955) 'Continuous convection from an instantaneous source of heat', *Q. J. R. Meteorol. 81*, 144–57, hereafter denoted P&B). This paper did not consider thermals or describe any new experiments, but the authors did derive plume equations that are very similar to those of MTT given above, and obtained very similar results. The authors did not make the entrainment assumption, but instead assumed that the shear stress between the plume and the environment is proportional to the square of the mean velocity $\overline{w}$, a parallel assumption based on dynamical similarity. They also considered more general initial source conditions, with finite source radius. At the end of MTT, the authors make some comparison between their solutions and those of P&B: in neutral conditions, the MTT solution (Eqn 2) has infinite velocity at the source, which then decreases, whereas the P&B solution has finite velocity at the source (of finite radius), which may then increase depending on the balance between buoyancy and momentum. However, the P&B solution gives a uniform spreading rate of the plume in stratified environments at all heights, which is unrealistic, at least when near the neutral density level.

For several years, controversy existed as to which was the more appropriate of these two models. A study by Fox (1970) utilised a form of the P&B model, which stimulated Morton to make a more detailed comparison between MTT, P&B and Fox's model (Morton 1971), and he concluded that MTT was preferable, again because it described more realistic behaviour at the upper levels. Priestley (1959, page 79) pointed out that the P&B model assumptions implied that in the P&B model the first of equations (1) took the more complex form

$$\frac{d\left(b^2\overline{w}\right)}{dz} = 2\alpha_1 b\overline{w} + \alpha_2 b^2\frac{d\overline{w}}{dz}, \qquad \text{...(3)}$$

'recognising that entrainment in a plume should be acceleration-dependent as well as velocity-dependent'. In the end, the MTT model won out largely because of the simplicity of the entrainment assumption, which was validated (in part) by experiments as described below, and also because Bruce Morton continued the development of the MTT model to more general situations. Priestley, however, always felt that the P&B model did not get the recognition it deserved. This has other parallels. Bill Priestley, being chief of CSIRO Division of Meteorological Physics (which later became the Division of Atmospheric Research, and is now part of CSIRO Marine and Atmospheric Research), wrote a short book (Priestley 1959, cited above) about the theoretical and experimental work of this laboratory on atmospheric boundary-layer turbulence and convection, much of which he had contributed himself. However, these accomplishments were later overshadowed by the results of the Russian group, notably Monin and Obukhov, whose work became more widely known in the 1960s. This included the development of the Monin-Obukhov length scale which became the basis for scaling the turbulent atmospheric boundary layer, and displaced results such as 'Deacon's law' (a power law relationship for wind profiles in the atmospheric boundary layer, proposed by Len Deacon of CSIRO Aspendale, but now discarded). But that's how science goes.

Forced plumes

Morton followed up MTT with two papers on 'forced plumes' (Morton 1959a, 1959b), which extended the MTT model to isolated sources with initial upward momentum with and without initial buoyancy, so that it covered turbulent jets as well as plumes in homogeneous or stratified environments. The first of these papers showed that all forced plumes from sources of finite radius may be related to that from a virtual point source at a lower level, and the second addressed some issues concerned with atmospheric pollution. A subsequent paper with a student, Jason Middleton (Morton and Middleton 1973), extended this mathematical formalism to show that all such flows from isolated sources with fluxes of mass (or volume) Q, momentum M and buoyancy (or heat) F may be characterised by a dimensionless parameter Γ, of the form

$$\Gamma = \frac{5FQ^2}{8\pi^{1/2}\alpha M^{5/2}}, \qquad \ldots(4)$$

where again α is the constant entrainment coefficient. This parameter has the character of a Richardson number, and assumes slightly different forms in different variations of the model (e.g. Boussinesq vs. non Boussinesq, top hat velocity and density profiles vs. Gaussian). Suitably defined, if $\Gamma = 1$ it characterises the flow as the pure plume MTT solution (Eqn 2). If $\Gamma < 0$, it represents negatively buoyant forced plumes with upward emission of fluid denser than the environment at the source—negatively buoyant jets, in effect. If $0 < \Gamma < 1$, the flows are positively buoyant jets with excess velocity relative to the 'pure' plume, and if $\Gamma > 1$ they represent 'lazy plumes', which have an initial velocity deficit. These lazy plumes have an initially decreasing radius if $\Gamma > 2.5$, but then accelerate to approach the asymptotic flow state (Eqn 2, if $N = 0$) as buoyancy 'kicks in'.

Some cartoons of the various possible types of flows without stratification are shown in Fig. 1. Figure 1(a) represents the 'pure plume' solution of equation (2) with $\Gamma = 1$, with small horizontal arrows indicating entrainment, and Fig. 1(b) represents the 'pure jet' solution with no buoyancy. Note the larger width of the jet compared with the buoyant plume. Figure 1(c) shows a depiction of a negatively buoyant jet ($\Gamma < 0$), and Fig. 1(c) shows a 'lazy plume', with buoyancy but little initial momentum. If background stratification is present, the flows with positive buoyancy end up in a form represented in Fig. 2, with lateral spreading at an equilibrium level.

Fig. 1. Schematic diagrams of various types of forced plumes in a homogeneous environment, as represented by the parameter Γ. (a) The 'pure plume' with $\Gamma = 1$, representing the original solution of Morton et al. 1956, which has some initial upward momentum as well as buoyancy; (b) the flow $\Gamma = 0$, with initial upward momentum but no buoyancy, representing a neutrally buoyant jet; (c) flow with $\Gamma < 0$, representing a collapsing column with upward initial momentum but negative buoyancy; (d) flows with $\Gamma > 1$ have positive buoyancy, but lack some initial momentum, and are known as 'lazy plumes'. If $\Gamma > 2.5$, they become narrower before ultimately spreading in the same manner as flows with $0 < \Gamma < 1$.

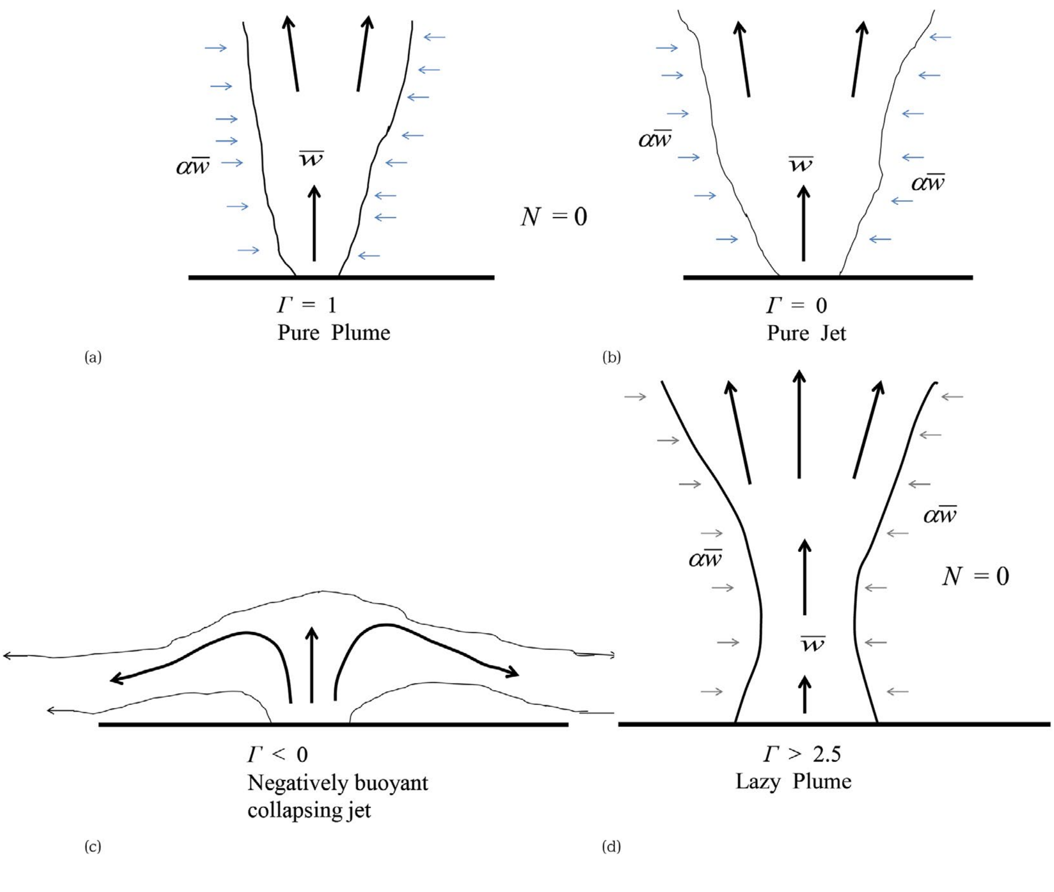


Fig. 2. This shows a representative flow with positive buoyancy, in which the fluid eventually reaches an equilibrium level of neutral density, where it spreads. The flow reaches a maximum height which scales with $(F_0/N^3)^{1/4}$.

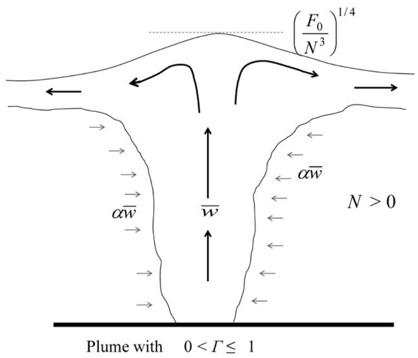
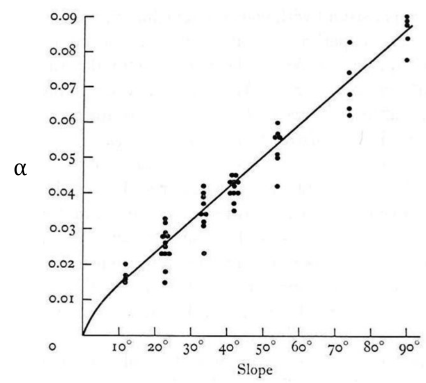


Fig. 3. Results from measurements of the entrainment coefficient α in flow down slopes into fresh water by Ellison and Turner (1959), as a function of the bottom slope angle. See also Turner (1973).



In recent years there has been considerable further development of this model, but still retaining the same basic assumptions. These include further studies of non-Boussinesq plumes, lazy plumes (e.g. Hunt and Kaye 2005), angled plumes, plumes in crossflow, plume-plume interactions, turbulent fountains (e.g. Bloomfield and Kerr 2000), and numerous others. A review of some of these developments and further references is given by Hunt and Van den Bremer (2010).

Plumes and water vapour

One area that has not been successfully pursued with these models concerns the effect of the inclusion of water vapour, with potential application to clouds. In an early paper on this topic, Morton (1957) added an extra equation for water vapour to the MTT model, to include the possible condensation and latent heat release of moisture. The analysis was limited to low levels (< 3 km) because the MTT model does not include the equation of state, but clouds form in the model in a stratified atmosphere if initial buoyancy flux is strong enough. He stated that the model may not be applicable if moisture processes were dominant. This was subsequently substantiated by Jack Warner of the cloud physics group of CSIRO Radiophysics Division (Warner 1970), who demonstrated that, when compared with observations of real clouds, steady-state models of clouds that incorporated continuous entrainment could not be adjusted to successfully describe both the cloud top height and the liquid water content. Further, observations (and even casual observations) of development of cumulus clouds showed that they did not spread horizontally as they grew vertically, but mostly had near-uniform width with height. This led to the concept of 'discontinuous, inhomogeneous entrainment' (implying that entrainment is large but intermittent and episodic) as a fundamental property of developing cumulus clouds (Baker 1980, Houze 1993), leading to much more complex models. In other words, the presence of latent energy in the cloud and environment introduces large scale turbulent structures that cannot be represented by models of the MTT type.

Volcanic plumes

Plumes with the largest initial buoyancy flux emanate from volcanic sources. Here the temperatures are of the order of 1000 K, the initial volume fluxes are massive for large eruptions, the ejected material may contain large amounts of hot solid material, and the columns may reach to great heights in the stratosphere. Models for these events need to be specially modified to include these factors. The most complete of such models of the MTT type is that developed by Woods (1988), as part of his Cambridge PhD thesis, which includes the equation of state for air, the emission of hot particles and the exchange of heat between solid particles and air. This paper also includes a comparison with previous models of volcanic plumes. Subsequent work (Woods and Bursik 1991) includes the effect of fallout of solid particles from the convecting column. A recent review of this work and application of these models to a variety of other different types of plumes occurring in nature is provided by Woods (2010).

Entrainment and detrainment

The MTT model makes the assumption that a steady rising turbulent plume entrains a horizontal inflow at a radially uniform velocity $u_e = \alpha\overline{w}$, where, as above, $\overline{w}$ is the mean vertical velocity in the plume, and α is the entrainment coefficient, with an assumed value of about 0.1. This is assumed to be due to turbulence within the plume on length scales that are smaller than the plume radius. How valid is this? Measurements of the entrainment coefficient were reported in MTT, based on experiments in which methylated spirits was released into salt water, in a tank about 1 m deep and 30 cm in diameter. From these and earlier experiments a value of α =0.093 was inferred, which is often rounded up to 0.1. In these flows there is always a distinct boundary between the non-turbulent fluid outside the plume, and the turbulent fluid inside it, with the latter containing large amounts of vorticity. Detailed observations of entrainment in convecting plumes in the laboratory are difficult to obtain because the finite size of the tank quickly becomes dominant. Observations of entrainment into two-dimensional wakes, jets and boundary layers (Townsend 1970, 1976) in larger facilities (such as wind tunnels) show ratios of $u_e / \overline{w}$ ranging from 0.024 to 0.21, depending on context. In general, most of the entrainment is associated with the largest eddies that occur in the laminar-turbulent boundary region, and appear as an 'engulfment' process. The observations suggest, in fact, that the dominant eddies have the length scale appropriate to those arising from linear instability of the mean velocity profile, which for a simple shear flow with coincident discontinuities in velocity and density would be Kelvin-Helmholtz instability.

Quantitative observations of entrainment into convecting fluids have been more revealing with flows down sloping surfaces. Ellison and Turner (1959) showed that the turbulent flow of (two-dimensional) dense fluid down a slope into a less-dense homogeneous environment entrained environmental fluid at a rate that depended on the Richardson number, defined by

$$Ri = \frac{g' h \cos\theta}{U^2}, \qquad \text{...(5)}$$

where θ is the slope angle relative to the horizontal, U the mean speed of the dense fluid (i.e. gravity current), h its thickness and $g' = g\Delta\rho/\rho$ denotes the reduced gravity as defined for Eqn (1) above. The entrainment coefficient is a function of Ri, but much of its dependence is due to variation of the slope angle θ, and the results for this are shown in Fig. 3. Note that as the slope angle tends to 90° so that the boundary becomes vertical, the entrainment coefficient α approaches 0.09, which is close to the value of 0.1 assumed for vertical axisymmetric plumes by MTT. Turner (1986) gives a detailed summary, but the observations from these (and similar) experiments are the best justification for this number, even to date, although the tanks used were not large and the Reynolds numbers achieved in many of the runs were modest.

Density-stratified environments

If the downslope flow is released into a density-stratified environment, a different set of phenomena may arise. When the environmental fluid is homogeneous, if any of the inflowing dense fluid is mixed with the environmental fluid, it is still heavy, and descends to the bottom of the tank with the dense unmixed fluid. However, if the environment is stratified, a portion of partially mixed fluid outside the main boundary of the descending plume may be free to find its own density level in the environment, and spread there, above the equilibrium level for the main downflow. The nett result is *detrainment* from the downflow, rather than *entrainment* into it. This is characterised by a new dimensionless parameter, B, termed the *Buoyancy number*, defined as (Baines 2001, 2002, 2005)

$$B = \frac{QN^3}{g'^2}, \qquad \text{...(6)}$$

where Q is the volume flux, g' the reduced gravity as above, and N the buoyancy frequency of the environment. A criterion for this new flow type depends on the drag coefficient C_D of the downflow on the sloping surface, and requires that

$$C_D \geq 0.2 B^{0.4} \sin\theta. \qquad \text{...(7)}$$

If this criterion is not met, the environmental fluid is entrained into the downflow much as for the homogeneous environment, until the equilibrium level is reached, where, as for plumes, it overshoots and settles back to spread laterally. If instead (7) is satisfied, wisps of fluid are detrained from the downflow by the energetic external eddies, until the downflow reaches its equilibrium level where again it may overshoot slightly, springback and spread.

Fig. 4. A cartoon showing the mixing processes occurring in flow of dense fluid down sufficiently steep slopes, into a density-stratified environment ($N > 0$). The physics is essentially the same as for flow into a homogeneous environment, with regions of large density and velocity gradient being approximately coincident (at the boundary of the downflow). This gives eddies of the Kelvin-Helmholz type at the boundary, resulting in entrainment.

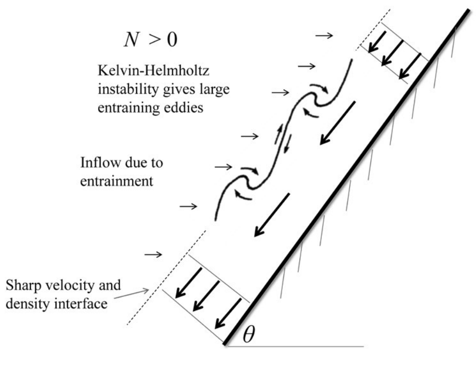
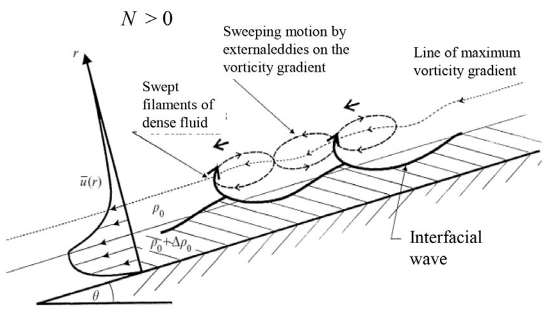


Fig. 5. Flow of dense fluid down a sufficiently gentle slope into a density-stratified environment, where bottom friction is important. Here the regions of large gradients in velocity and density are not coincident, giving eddies that are mostly outside (i.e. above) the dense fluid, with the structure suggested by Holmboe instability.



Which process occurs can be approximately described by linear stability theory. If the slope is very steep, the near discontinuities in velocity and density coincide, and the profile is subject to Kelvin-Helmholtz instability, with large turbulent eddies implying entrainment into the downflow until the level of neutral buoyancy is reached, much as for a homogeneous environment. A schematic cartoon of these dynamics is shown in Fig. 4. However, if the slope is small (say, < 20°) and the bottom drag is significant, the region of maximum mean velocity gradient lies above the density interface, and the system is subject to Holmboe instability (a form of instability in flows where the regions of varying velocity and varying density do not coincide—see for example Baines 1998, Chapter 4), resulting in eddies that are mostly above (i.e. outside) the downflow, sweeping wisps of dense fluid from the interface which then become mixed into the environment, as indicated in Fig. 5.

Both types of behaviour have been observed in the ocean. Entrainment is common, and occurs on the Denmark Strait overflow, increasing the transport of the Atlantic meridional overturning circulation. It also occurs on the Antarctic continental slope, where dense water flows off the continental shelf (in the Weddell and Ross Seas, and elsewhere) to form Antarctic bottom water, with entrainment contributing significantly to the total flux (Baines 2009). On the other hand, detrainment has been inferred from observations in the two deep channels carrying the outflow of salty fluid from the Red Sea (Baines 2008).

Conclusion

In summary, the study of the dynamics of plumes is just part of the broad field of study of turbulent flows in which buoyancy effects are important, which include the earth's atmosphere and oceans. The review of plume theory by Hunt and van den Bremer (2010) commences with a Russian paper by Zeldovich in 1937, but others might prefer to regard the study of this subject as beginning with G.I. Taylor's observations of convection at sea off the coast on Newfoundland, using kites and balloons from the ice-scout sailing ship *Scotia* (Taylor 1915), an expedition that was inspired by the loss of the *Titanic* to an iceberg in 1912. There may well be earlier, more deserving options. Whatever, Bruce Morton's work on plumes has been central to this field, which is still a very active area of study. New variations and applications of the type of model that he helped to initiate continue to be developed, as indicated by the accelerated rate of increase in the number of citations to his publications.

References

Baines, P.G. 1998. *Topographic effects in stratified flows*. Cambridge University Press, 482pp.

Baines, P.G. 2001. Mixing in flows down gentle slopes into stratified environments. *J. Fluid Mech., 443*, 237–70

Baines, P.G. 2002. Two-dimensional plumes in stratified environments. *J. Fluid Mech., 471*, 315–37.

Baines, P.G. 2005. Mixing regimes for the flow of dense fluid down slopes into stratified environments. *J. Fluid Mech., 538*, 245–67.

Baines, P.G. 2008. Mixing in downslope flows in the ocean—plumes versus gravity currents. *Atmosphere-Ocean, 46*, 405–19, doi:10.3137/AO925.2008.

Baines, P.G. 2009. A model for the structure of the Antarctic Slope Front. *Deep-Sea Research II, 56*, 859–73.

Baker, M.B., Corbin, R.G., and Latham, J. 1980. The influence of entrainment on the evolution of cloud droplet spectra. I: A model in inhomogeneous mixing. *Quart. J Roy. Met. Soc., 106*, 581–98.

Bloomfield, L.J. and Kerr, R. C. 2000. A theoretical model of a turbulent fountain. *J. Fluid Mech., 424*, 197–206.

Ellison, T.H. and Turner, J.S. 1959. Turbulent entrainment in stratified flows. *J. Fluid Mech., 6*, 423-448.

Fox, D.G. 1970. Forced plume in a stably stratified fluid. *J. Geophys. Res., 75*, 6818–35.

Houze, R.A. 1993. *Cloud Dynamics*, Academic Press, San Diego, 573pp.

Hunt, G.R. and Kaye, N.B. 2005. Lazy plumes. *J. Fluid Mech., 533*, 329–38.

Hunt, G.R. and van den Bremer, T.S. 2011. Classical plume theory: 1937-2010 and beyond. *IMA Journal of Applied Math., 76*, 424–48.

Morton, B.R. 1957. Buoyant plumes in a moist atmosphere. *J. Fluid Mech., 2*, 127–44.

Morton, B.R. 1959a. Forced plumes. *J. Fluid Mech., 5*, 151-163.

Morton, B.R. 1959b. The ascent of turbulent forced plumes in a calm atmosphere. *Int. J. Air Poll., 1*, 184–97.

Morton, B.R. 1971. The choice of conservation equations for plume models. *J. Geophys. Res., 76*, 7409-–16.

Morton, B.R. 1984 The generation and decay of vorticity. Geophys. Astrophys. *Fluid Dyn. 28*, 277-308.

Morton, B.R. and Middleton, J. 1973. Scale diagrams for forced plumes. *J. Fluid Mech., 58*, 165–76.

Morton, B.R., Taylor, G.I. and Turner, J.S. 1956. Turbulent gravitational convection from maintained and instantaneous sources, *Proc. Roy. Soc. A, 234*, 1–23.

Priestley, C.H.B. and Ball, F.K. 1955. Continuous convection from an instantaneous source of heat, *Quart. J. Roy. Met .Soc., 81*, 144–57.

Priestley, C.H.B. 1959. *Turbulent transfer in the lower atmosphere*, University of Chicago Press, 130pp.

Saffman, P.G. 1992. *Vortex dynamics*. Cambridge University Press, 311pp.

Sparks, R.S.J., Bursik, M., Carey, S., Gilbert, J., Sigurdson, H., and Woods, A.W. 1997. *Volcanic plumes,* Wiley, New York.

Taylor, G.I. 1915. Eddy motion in the atmosphere. *Philos. Trans. R. Soc. A, 215,* 1–26.

Townsend, A.A. 1970. Entrainment and the structure of turbulent flow. *J. Fluid Mech., 41*, 13–46.

Townsend, A.A. 1976. *The structure of turbulent shear flow*, 2nd edition, Cambridge University Press, 367pp.

Turner, J.S. 1973. *Buoyancy effects in fluids*. Cambridge University Press, 367pp.

Turner, J.S. 1986. Turbulent entrainment: the development of the entrainment assumption, and its application to geophysical flows. *J. Fluid Mech., 173*, 431–71.

Warner, J. 1970. On steady-state models of one-dimensional convection. *J. Atmos. Sci., 27*, 1035–40.

Woods, A.W. 1988. The dynamics and thermodynamics of eruption columns. *Bull. Volcanol., 50*, 169–93.

Woods, A.W. 2010. Turbulent plumes in nature. *Ann. Rev. Fluid Mech., 42*, 391–412.

Woods, A.W. and Bursik, M. 1991. Particle fallout, thermal disequilibrium and volcanic plumes. *Bull. Volcanol., 53*, 559–70.